Topology and Functional Analysis

Unit 1: Set Theory and Logic - Topological spaces - Closed sets and limit points.

Unit 2: Continuous functions - product topology - metric topology - Quotient topology.

Unit 3: Connnectedness and compactness.

Unit 4: The countability Axioms - The separation Axiom - Normal spaces - The Urysohn Lemma.

Unit 5: Banach Spaces.

Unit 1

0.1 Topological Spaces

Definition 0.1.1. A *topology* on a set X is a collection $\mathcal{J}$ of subsets of X having the following properties:

(i)$\emptyset$ and X are in $\mathcal{J}$.

(ii) The union of the elements of any subcollection of $\mathcal{J}$ is in $\mathcal{J}$.

(iii) The intersection of the elements of any finite subcollection of $\mathcal{J}$ is in $\mathcal{J}$.

A set X for which a topology $\mathcal{J}$ has been specified is called a *topological space.*

If X is a topological space with topology $\mathcal{J}$, we say that a subset U of X is an *open set* of X. If U belongs to the collection $\mathcal{J}$.

If X is any set, the collection of all subsets of X is a topology on X, it is called the *discrete topology.* The collection consisting of X and $\emptyset$ only is also a topology on X, it is called the *indiscrete topology* or the *trivial topology.*

Let X be a set. Let $\mathcal{J}_f$ be a collection of all subsets U of X such that $X-U$ either is finite or is all of X. Then $\mathcal{J}_f$ is a topology on X, called the *finite complement topology.*

Result 0.1.2. *$\mathcal{J}_f$ is a finite complement topology.*

Proof. Since $X - X = \emptyset$ and $X - \emptyset = X$, either is finite or is all of X.

Both X and $\emptyset$ are in $\mathcal{J}_f$.

To show that $\bigcup U_\alpha$ is in $\mathcal{J}_f$.

$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$.

Since $X - U_\alpha$ is finite then $\bigcap (X - U_\alpha)$ is finite.

Then $(X - \bigcup U_\alpha)$ is finite.

Therefore, $\bigcup U_\alpha$ is in $\mathcal{J}_f$.

If $U_1, U_2, \cdots, U_n$ or non empty elements of $\mathcal{J}_f$.

To show that $\bigcap U_i$ is in $\mathcal{J}_f$.

Now we know that $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$.

since $(X - U_i)$ is finite then $\bigcup_{i=1}^{n} (X - U_i)$ is finite.

Then $\bigcap U_\alpha$ is in $\mathcal{J}_f$.

Therefore, $\mathcal{J}_f$ is a finite complement topology. □

Definition 0.1.3. Suppose that $\mathcal{J}$ and $\mathcal{J}'$ are two topologies on a given set X. If $\mathcal{J}' \supset \mathcal{J}$, we say that $\mathcal{J}'$ is *finer* than $\mathcal{J}$; if $\mathcal{J}'$ properly contains $\mathcal{J}$, we say that $\mathcal{J}'$ is *strictly finer* than $\mathcal{J}$. We also say that $\mathcal{J}$ is *coarser* than $\mathcal{J}'$, or *strictly coarser*, in these two respective situations. We say $\mathcal{J}$ is *comparable* with $\mathcal{J}'$ if either $\mathcal{J}' \supset \mathcal{J}$ or $\mathcal{J} \supset \mathcal{J}'$.

0.2 Basis for a Topology

Definition 0.2.1. If X is a set, a *basis* for a topology on X is a collection $\mathscr{B}$ of subsets of X (called *basis elements*) such that

(i) For each $x \in X$, there is at least one basis element B containing x.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathscr{B}$ satisfies these two conditions, then we define the *topology* $\mathcal{J}$ *generated by* $\mathscr{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{J}$) if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{J}$.

Lemma 0.2.2. *Let X be a set; let $\mathscr{B}$ be a basis for a topology $\mathcal{J}$ on X. Then $\mathcal{J}$ equals the collection of all unions of elements of $\mathscr{B}$.*

Proof. Let X be a set and $\mathscr{B}$ be the basis for the topology $\mathcal{J}$ on X.
The collection of elements of $\mathscr{B}$ are also elements of $\mathcal{J}$ because $\mathcal{J}$ is a topology, their union is in $\mathcal{J}$.
Conversely, given $U \in \mathcal{J}$, choose for each $x \in U$ an element B_x of $\mathscr{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathscr{B}$. □

Lemma 0.2.3. *Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology of X.*

Proof. First we prove that $\mathscr{C}$ is a basis.
Given $x \in X$, since X is an open set, by hypothesis an element C of $\mathscr{C}$ such that $x \in C \subset X$.
Let $x \in C_1 \cap C_2$ where C_1 and C_2 are the elements of $\mathscr{C}$.
Since C_1 and C_2 are open, $C_1 \cap C_2$ are open.
By hyphothesis, there exists an element C_3 of $\mathscr{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
Therefore, $\mathscr{C}$ is a basis.
Let $\mathcal{J}$ be the topology on X.
Let $\mathcal{J}'$ denote the topology generated by $\mathscr{C}$.
To prove that $\mathcal{J}' = \mathcal{J}$.
By 0.2.4, $\mathcal{J}'$ is finer than $\mathcal{J}$.
Conversely, since each element of $\mathscr{C}$ is an element of $\mathcal{J}$, the union of elements of $\mathscr{C}$ is also in $\mathcal{J}$.
By 0.2.2, $\mathcal{J}'$ contains $\mathcal{J}$.

Therefore, $\mathcal{J}' = \mathcal{J}$.

Therefore, $\mathscr{C}$ is a basis for the topology of X. $\square$

Lemma 0.2.4. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{J}$ and $\mathcal{J}'$, respectively, on X. Then the following are equivalent:*

(i) $\mathcal{J}'$ is finer than $\mathcal{J}$.

(ii) For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof. To prove (ii)⇒(i)

Given an element $U \in \mathcal{J}$.

To show that $U \in \mathcal{J}'$.

Let $x \in U$. Since $\mathscr{B}$ generates $\mathcal{J}$, there is an element $B \in \mathscr{B}$ such that $x \in B \subset U$.

By (ii), there exists an element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$, then $x \in B' \subset U$.

By definition of basis for the topology, $U \in \mathcal{J}'$.

To prove (i)⇒(ii)

Given $x \in X$ and $B \in \mathscr{B}$ with $x \in B$.

Now $B \in \mathcal{J}$, by definition and $\mathcal{J} \subset \mathcal{J}'$ by (i); therefore $B \in \mathcal{J}'$.

Since $\mathcal{J}'$ is generated by $\mathscr{B}'$, there is an element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

$\square$

Definition 0.2.5. If $\mathscr{B}$ is the collection of all open intervals in the real line,

$$(a, b) = \{x | a < x < b\},$$

the topology generated by $\mathscr{B}$ is called the *standard topology* on the real line.

If $\mathscr{B}'$ is the collection of all half-open intervals of the form

$$[a, b) = \{x | a \leq x < b\},$$

where $a < b$, the topology generated by $\mathscr{B}'$ is called the *lower limit topology* on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$. Finally let K denote the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let $\mathscr{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) - K$. The topology generated by $\mathscr{B}''$ will be called the *K-topology* on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_k$.

Lemma 0.2.6. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are strictly finer than three standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{J}, \mathcal{J}', \mathcal{J}''$ be the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_k$,respectively.
Given a basis element (a, b) for $\mathcal{J}$ and a point x of (a, b), the basis element $[x, b)$ for $\mathcal{J}'$ contains x and lies in (a, b). On the otherhand,given the basis element $[x, d)$ for $\mathcal{J}'$, there is no open interval (a, b) that contains x and lies in $[x, d)$. Thus $\mathcal{J}'$ is strictly finer than $\mathcal{J}$.
Given a basis element (a, b) for $\mathcal{J}$ and a point x of (a, b), this same interval is a basis element for $\mathcal{J}''$ that contains x. On the otherhand, given the basis element $B = (-1, 1) - K$ for $\mathcal{J}''$ and the point O of B, there is no open interval that contains O and lies in B.
By definition of comparable, $\mathcal{J}'$ and $\mathcal{J}''$ are not comparable with one another. □

Definition 0.2.7. *A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to* *be the collection $\mathcal{J}$ of all unions of finite intersections of elements of $\mathcal{S}$.*

0.3 The Order Topology

Definition 0.3.1. If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the *order topology.*

Suppose that X is a set having a simple order relation $<$. Given elements a and b of X such that $a < b$, there are four subsets of X that are called the *intervals* determined by a and b. They are the following:

$$(a, b) = \{x | a < x < b\},$$
$$(a, b] = \{x | a < x \leq b\},$$
$$[a, b) = \{x | a \leq x < b\},$$
$$[a, b] = \{x | a \leq x \leq b\}.$$

A set of the first type is called an *open interval* in X, a set of the last type is called a *closed interval* in X, and sets of the second and third types are called *half-open intervals.*

Definition 0.3.2. Let X be a set with a simple order relation; assume X has more than one element. Let $\mathscr{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element(if any) of X.

The collection $\mathscr{B}$ is a basis for a topology on X, which is called the *order topology.*

Definition 0.3.3. If X is an ordered set, and a is an element of X, there are four subsets of X that are called *rays* determined by a. They are the following:

$$\begin{aligned}(a, +\infty) &= \{x|x > a\},\\ (-\infty, a) &= \{x|x < a\},\\ [a, +\infty) &= \{x|x \geq a\},\\ (-\infty, a] &= \{x|x \leq a\}.\end{aligned}$$

Sets of the first types are called *open rays*, and sets of the last two types are called *closed rays.*

0.4 The product Topology on $X \times Y$

Definition 0.4.1. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 0.4.2. *If $\mathscr{B}$ is a basis for the topology of X and $\mathscr{C}$ is a basis for the topology of Y, then the collection*

$$\mathscr{D} = \{B \times C | B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. We apply 0.2.3. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that

$x \times y \in U \times V \subset W$.

Because $\mathscr{B}$ and $\mathscr{C}$ are bases for X and Y respectively, we can choose an element B of $\mathscr{B}$ such that $x \in B \subset U$ and an element C of $\mathscr{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$.

Therefore, $\mathscr{D}$ is a basis for $X \times Y$. □

Definition 0.4.3. Let $\pi_1 : X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

We use the word "onto" because π_1 and π_2 are surjective.

Note If U is an open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then

$$\pi_2^{-1}(V) = X \times V,$$

which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$.

Theorem 0.4.4. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U)|U \ open \ in \ X\} \cup \{\pi_2^{-1}(V)|V \ open \ in \ Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{J}$ denote the product topology on $X \times Y$.

Let $\mathcal{J}'$ be the topology generated by $\mathcal{S}$. Because every element of $\mathcal{S}$ belongs to $\mathcal{J}$.

By definition of subbasis, arbitrary unions of finite intersections of elements of $\mathcal{S}$.

Thus $\mathcal{J}' \subset \mathcal{J}$.

On the otherhand,

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

where $\pi_1^{-1}(U)$ is open in X and $\pi_2^{-1}(V)$ is open in Y.

Since $U \times V \in \mathcal{J}$, we have $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$. $U \times V \in \mathcal{J}'$. Therefore, $\mathcal{J} \subset \mathcal{J}'$. □

0.5 The Subspace Topology

Definition 0.5.1. Let X be a topological space with topology $\mathcal{J}$. If Y is a subset of X, the collection

$$\mathcal{J}_Y = \{Y \cap U | U \in \mathcal{J}\}$$

is a topology on Y, called the *subspace topology.* With this topology, Y is called a *subspace* of X; its open sets consist of all intersections of open sets of X with Y.

Lemma 0.5.2. *If $\mathscr{B}$ is a basis for the topology of X then the collection*

$$\mathscr{B}_Y = \{B \cap Y | B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

Proof. Consider U is open in X. Given $\mathscr{B}$ is a basis for the topology of X. We can choose an element B of $\mathscr{B}$ such that $y \in B \subset U$.
Then $y \in B \cap Y \subset U \cap Y$, since $\mathscr{B}_Y = \{B \cap Y | B \in \mathscr{B}\}$.
By 0.2.3 or definition of basis, $\mathscr{B}_Y$ is a basis for the subspace topology on Y. □

Definition 0.5.3. If Y is a subspace of X, we say that a set U is *open in* Y(or open *relative* to Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is *open in* X if it belongs to the topology of X.

Lemma 0.5.4. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Proof. Given U is open in Y and Y is open in X.
Since U is open in Y and Y is a subspace of X then $U = Y \cap V$ where V is open in X.
Since Y and V are both open in X, $Y \cap V$ is open in X.
Therefore, U is open in X. □

Theorem 0.5.5. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y.

Then $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and$B$ respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product on $A \times B$.

The bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the topologies are the same. □

Theorem 0.5.6. *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Consider the ray $(a, +\infty)$ in X.

If $a \in Y$, then $(a, +\infty) \cap Y = \{x | x \in Y \text{ and } x > a\}$; this is an open ray of the ordered set Y.

If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y, since Y is convex.

If $a \in Y$, the set $(a, +\infty) \cap Y$ equals all of Y. If $a \notin Y$, it is empty.

Similarly the intersection of the ray $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself or empty.

Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y and since each is open in the order topology, the order topology contains the subspace topology.

Conversely, Y equals the intersection of X with Y, that is $X \cap Y = Y$. So it is open in the subspace topology on Y. The order topology is contained in the subspace topology. Therefore, the order topology and subspace topology are same. □

0.6 Closed Sets and Limit Points

Definition 0.6.1. A subset A of a topological space X is said to be *closed* if the set $X - A$ is open.

Theorem 0.6.2. *Let X be a topological space. Then the following conditions hold:*
(1) $\emptyset$ and X are closed.
(2) Arbitrary intersections of closed sets are closed.
(3) Finite unions of closed sets are closed.

Proof. (1) $\emptyset$ and X are closed because they are the complements of the open set X and $\emptyset$ respectively.
(2) Consider a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

Since the sets $X - A_\alpha$ are open. By definition of closed sets, the right side of this equation represents an arbitrary union of open sets and is thus open. Therefore, $\bigcap A_\alpha$ is closed.

(3) Similarly, if A_i is closed for $i = 1, 2, \cdots, n$. Consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$$

The set on the right side of this equation is a finte intersection of open sets and is therefore open. Hence $\bigcup A_i$ is closed. □

Definition 0.6.3. If Y is a subspace of X, we say that a set A is *closed in* Y if A is a subset of Y and if A is closed in the subspace topology of Y(that is, if $Y - A$ is open in Y).

Theorem 0.6.4. *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y. By the definition of the subspace topology, but $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. Then $Y - A$ is open in Y. By definition, it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in X and $A = Y \cap (X - U)$. Hence A equals the intersection of a closed set of X with Y. □

Theorem 0.6.5. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. Given A is closed in Y and Y is closed in X. Since A is closed in Y and Y is a subspace of X.
Let $A = Y \cap (X - B)$ where $X - B$ is open in X. Then B is closed in X. Since

Y and B are both closed in X. Then $Y \cap (X - B)$ is closed in X. Therefore, A is closed in X. □

Definition 0.6.6. Given a subset A of a topological space X, the *interior* of A is defined as the union of all open sets contained in A, and the *closure* of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by Int A and the closure of A is denoted by Cl A or by $\overline{A}$. Obviously Int A is an open set and $\overline{A}$ is a closed set; furthermore,

$$\text{Int } A \subset A \subset \overline{A}.$$

If A is open, A=Int A; while if A is closed, $A = \overline{A}$.

Theorem 0.6.7. *Let Y be a subspace of X; let A be a subset of Y; let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. The set $\overline{A}$ is closed in X, so $\overline{A} \cap Y$ is closed in Y. By 0.6.4, since $\overline{A} \cap Y$ contains A and since B is closed. By definition B equals the intersection of all closed subsets of Y containing A, we must have $B \cap (\overline{A} \cap Y)$.

On the otherhand, we know that B is closed in Y. By 0.6.4, $B = C \cap Y$ for some set C closed in X. Then C is a closed set of X containing A; because $\overline{A}$ is the intersection of all such closed sets, we conclude that $\overline{A} \subset C$. Then $(\overline{A} \cap Y) \subset (C \cap Y) = B$. Therefore, $B = \overline{A} \cap Y$. □

Theorem 0.6.8. *Let A be a subset of the topological space X.*

(a) Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof. (a)We prove this theorem by contrapositive method.
If x is not in A, since A is closed, $A = \overline{A}$. The set $U = X - A$ is an open set containing x that does not intersect A.
Conversely, if there exists an open set U containing x which does not intersect A. Then $X - U$ is a closed set containing A.
By definition of the closure $\overline{A}$, the set $X - U$ must contain $\overline{A}$, since $x \in U$. Therefore, x cannot be in $\overline{A}$.
(b) Write the definition of topology generated by basis,if every open set x intersects A, so does every basis element B containing x, because B is an open set.
Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x. □

Definition 0.6.9. If A is a subset of the topological space X and if x is a point of X, we say that x is a *limit point*(or "cluster point" or "point of accumulation") of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Theorem 0.6.10. *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Proof. Let A' be the set of all limit points of A.
If $x \in A'$, every neighborhood of x intersects of A in a point different from x. By 0.6.8, $x \in \overline{A}$. Then $A' \subset \overline{A}$.
By definition of closure, $A \subset \overline{A}$. Therefore, $A \cup A' \subset \overline{A}$.

Conversely, let $x \in \overline{A}$

To show that $\overline{A} \subset A \cup A'$

If $x \in A$ then it is trivially true for $x \in A \cup A'$.

Suppose $x \notin A$. Since $x \in \overline{A}$, by 0.6.8, we know that every neighborhood U of x intersect A, because $x \notin A$, the set U must intersect A in a point different from x. Then $x \in A'$ so that $x \in A \cup A'$.

Then $\overline{A} \subset A \cup A'$.

Therefore, $A = A \cup A'$. □

Corollary 0.6.11. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. The set A is closed iff $A = \overline{A}$. By 0.6.10, $A' \subset A$. □

Definition 0.6.12. A topological space X is called a *Hausdroff space* if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.

Theorem 0.6.13. *Every finite point set in a Hausdorff space X is closed.*

Proof. It is enough to show that every one-point set $\{x_0\}$ is closed.

If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V respectively.

Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$.

As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself.

Therefore, $\{x_0\}$ is closed. □

Note: The condition that finite point sets be closed is in fact weaker than the Hausdroff condition. For example, the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed has been given a name of its own; it is called the T_1 *axiom*.

Theorem 0.6.14. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A.
Conversely, suppose that x is a limit point of A and suppose some neighborhood U of x intersects A in only finitely many points.
Let $\{x_1, x_2, \cdots, x_m\}$ be the points of $U \cap (A - \{x\})$.
The set $X - \{x_1, x_2, \cdots, x_m\}$ is an open set of X, since the finite point set $\{x_1, x_2, \cdots, x_m\}$ is closed then

$$U \cap (X - \{x_1, x_2, \cdots, x_m\})$$

is a neighborhood of x that does not intersects the set $A - \{x\}$. Since $\{x_1, x_2, \cdots, x_m\}$ be points of $U \cap (A - \{x\})$.
This contradicts the assumption that x is a limit point of A. □

Theorem 0.6.15. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose that x_n is a sequence of points of X that converges to x.

If $y \neq x$, let U and V be disjoint neighborhoods of x and y respectively. Since U contains x_n for all but finitely many values of n, the set V cannot contains x_n. Therefore,x_n cannot converge.

If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$.

Therefore, x is the limit of the sequence x_n. □

Theorem 0.6.16. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Proof. Let X and Y be two Hausdorff spaces.

To prove $X \times Y$ is Hausdorff.

Let $x_1 \times y_1$ and $x_2 \times y_2$ be two distinct points of $X \times Y$. Then x_1, x_2 are distinct points of X and X is a Hausdorff space, there exists neighborhood U_1 and U_2 of x_1 and x_2 such that $U_1 \cap U_2 = \emptyset$

Similarly, y_1, y_2 are distinct point of Y and Y is a Hausdorff space, there exists neighborhood V_1 and V_2 of y_1 and y_2 such that $V_1 \cap V_2 = \emptyset$.

Then clearly $U_1 \times V_1$ and $U_2 \times V_2$ are open sets in $X \times Y$ containing $x_1 \times y_1$ and $x_2 \times y_2$ such that $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$.

Therefore, $X \times Y$ is a Hausdorff space.

Let X be a Hausdorff space and let Y be a subspace.

To prove Y is a Hausdorff space.

Let y_1, y_2 be two distinct points of Y and Y containing X. Then y_1 and y_2 are distinct points in X and X is Hausdorff there exists neighborhood U_1 and U_2 of y_1 and y_2 such that $U_1 \cap U_2 = \emptyset$. Then $U_1 \cap Y$ and $U_2 \cap Y$ are distinct neighborhoods of y_1 and y_2 in Y.

Therefore, Y is a Hausdorff space. □

Unit 2

0.7 Continuity of a Function

Definition 0.7.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

$f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set $f(X)$ of f.

Theorem 0.7.2. *Let X and Y be the topological spaces.Let $f : X \to Y$. Then the following are equivalent:*

(a) f is continuous.

(b) For every subset of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(c) For every closed set B of Y, a set $f^{-1}(B)$ is closed in X.

(d) For each $x \in X$ and each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$.

If the condition in equation (d) holds for the point x of X such that f is continuous at the point x.

Proof. To show that (a)$\Rightarrow (b) \Rightarrow (c) \Rightarrow$(a) and (a)$\Rightarrow (d), (d) \Rightarrow (a)$.

First we show that (a)$\Rightarrow$(b)

Assume f is continuous. Let A be a subset of X. We have to show that $f(\overline{A}) \subset \overline{f(A)}$.

If $x \in \overline{A}$ then $f(x) \in f(\overline{A})$. Since f is continuous, $f^{-1}(V)$ is an open set of X containing x, where V be a neighborhood of $f(x)$.

Now $f^{-1}(V)$ must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, $f(x) \in \overline{f(A)}$. Therefore, $f(\overline{A}) \subset \overline{f(A)}$.

To show that (b)$\Rightarrow$(C)

Let B be closed in Y. Let $A = f^{-1}(B)$.

To prove that A is closed in X.

ie, To prove that $\overline{A} = A$.

By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$

If $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$.

Then $x \in f^{-1}(B) \Rightarrow x \in A$. Therefore, $\overline{A} \subset A$.

Since $A \subset \overline{A}$, therefore, $\overline{A} = A$.

To show that (c)$\Rightarrow$(a)

Let V be open in Y. The set $B = Y - V$.

Then $f^{-1}(B) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$

Now B is a closed set of Y then $f^{-1}(B)$ is closed in X(By hypothesis).

Then $f^{-1}(V)$ is open in X.

Therefore, f is continuous.

To show that (a)$\Rightarrow$(d)

Let $x \in X$. Let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x.

Therefore, $f(U) \subset V$.

To show that (d)$\Rightarrow$(a)

Let V be open in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

Then by hypothesis, there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$.

Now $f^{-1}(V)$ can be written as the union of the open sets U_x.

Thus $f^{-1}(V)$ is open.

Therefore, f is continuous. $\square$

Definition 0.7.3. Let X and Y be topological spaces. Let $f : X \to Y$ be a bijection. If both the function x and the inverse function $f^{-1}(V)$ are continuous then f is called homeomorphism.

Theorem 0.7.4. *(Rules for constructing continuous functions). Let X, Y and Z be topological spaces.*

(a) (constant function) If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is a continuous. Let A is a subspace of X. Then the restricted function $f/A : A \to Y$ is continuous.

(e) (Restricting or expanding the range) Let $f : X \to Y$ be a continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous.

If Z is a space having Y as a subspace then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous, if X can be written as the union of open set U_α such that f/U_α is continuous for each α.

Proof. (a) Let $f(x) = y_0, x \in X, y_0 \in Y$.

Let V be open in Y.

If $y_0 \in Y$, the set $f^{-1}(V) = X$.

The set $f^{-1}(V)$ be open in $X, y_0 \subset V$

Therefore, f is continuous.

(b) Let A be a subspace of X. To prove $j : A \to X$ is continuous.

If U is open in X then $j^{-1}(U) = U \cap A$ which is open in A by definition of subspace topology.

Then $j^{-1}(U)$ is open in A.

Therefore, j is continuous.

(c) Since f and g be continuous. We have the following conditions:

If U is open in Z then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.

Then $(g \circ f)^{-1}(U)$ is open in X. Therefore, $g \circ f : X \to Z$ is continuous.

(d)Let $f : X \to Y$ be continuous. Let A be a subspace of X.

To prove $f/A : A \to Y$ is continuous.

Since by (b), we have the inclusion map $j : A \to X$ is continuous. Also we have $f : X \to Y$ is continuous.

Therefore, the restricted function $f/A : A \to Y$ is continuous by (c).

ie, f/A each equals the composite of the inclusion map j.

(e) Let $f : X \to Y$ is continuous.

Given Z is a subspace of Y containing the image set $f(X)$. ie,$f(X) \subset Z \subset Y$

To prove the function $g : X \to Z$ obtained from f is continuous.

Let B be open in Z. Since Z is a subspace of $Y, B = Z \cap U$ for some open set U of Y.

Since B is open in Z, $g^{-1}(B)$ is open in X and since U is open in Y, $f^{-1}(U)$ is open in X

Then $f^{-1}(U) = g^{-1}(B)$

Therefore, $g : X \to Z$ obtained from f is continuous.

If Z is a space having Y as a subspace. To prove the function $h : X \to Z$ is continuous.

This is obtained by the composition of the map $f : X \to Y$ and the inclusion map $j : Y \to Z$.

Since Y is a subspace of Z, inclusion map $j : Y \to Z$ is continuous by (b).

Therefore, the function $h : X \to Z$ is continuous.

(f) Given X can be written as the union of open sets U_α such that f/U_α is continuous for each α.

To prove $f : X \to Y$ is continuous.

Let V be open in Y.

Now $f(x) \in V, x \in X$. Since U_α is open in X containing x. Then $f^{-1}(V) \cap U_\alpha$ is open in X.

Since f/U_α is continuous; U_α is open in $X, (f/U_\alpha)^{-1}(V)$ is open in X.

Then $f^{-1}(V)$ is open in X.

Therefore, f is continuous. □

Theorem 0.7.5. *(The Pasting Lemma) Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y, B$ is continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.*

Proof. Let $X = A \cup B$ where A and B are closed in X.

Since $f : A \to Y$ is continuous, $f^{-1}(C)$ is closed in A, where C is closed in Y.

Since $g : B \to Y$ is continuous, $g^{-1}(C)$ is closed in B where C is closed in Y.

If $x \in A, h(x) = f(x)$ and if $x \in B, h(x) = g(x)$.

If $x \in A \cup B, h(x) = f(x) \cup g(x)$.

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$.

Then $h^{-1}(C)$ is closed in $A \cup B$.

Then $h^{-1}(C)$ is closed in X.

Therefore, h is continuous. □

Theorem 0.7.6. *(Maps into products) Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \text{ and } f_2 : A \to Y$$

are continuous.

The maps f_1 and f_2 are called the coordinate functions of f.

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto its first and second factors. These maps are continuous..

For, $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

If U and V are open, these sets are open.

Since $f : A \to X \times Y, \pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, for every $a \in A$.

Since $f_1 : A \to X$ and $f_2 : A \to Y$

$$f_1(a) = \pi_1(f(a)) \text{ and } f_2(a) = \pi_2(f(a))$$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions, f_1 and f_2 are continuous.

Conversely, suppose f_1 and f_2 are continuous. Then $f_1^{-1}(U)$ is open in A and $f_2^{-1}(V)$ is open in A.

$$a \in f_1^{-1}(U) \cap f_2^{-1}(V)$$

Also we have $U \times V$ be the basis element for the topology on $X \times Y$ then

$$\begin{gathered} f(a) \in U \times V \\ \Rightarrow a \in f^{-1}(U \times V) \\ \Rightarrow f_1^{-1}(U) \cap f_2^{-1}(V) \subset f^{-1}(U \times V) \end{gathered}$$

Also if $a \in f^{-1}(U \times V) \Rightarrow f(a) \in U \times V$

$$\begin{gathered} \Rightarrow (f_1(a), f_2(a)) \in U \times V \\ \Rightarrow f_1(a) \in U, f_2(a) \in V \\ \Rightarrow a \in f^{-1}(U), a \in f_2^{-1}(V) \\ f^{-1}(U \times V) \subset f_1^{-1}(U) \cap f_2^{-1}(V) \\ f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V) \end{gathered}$$

Since $f_1^{-1}(U)$ and $f_2^{-1}(V)$ is open in A.

Then $f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in A.

Then $f^{-1}(U \times V)$ is open in A.

Therefore, f is continuous. $\square$

0.8 The Product Topology

Definition 0.8.1. Let J be an index set. Given a set X, we define *J-tuple* of elements of X to be a function $\mathbf{x} : J \to X$. If α is an element of J, we often denote the value of $\mathbf{x}$ at α by x_α rather than $\mathbf{x}(\alpha)$; we call it the αth *coordinate* of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_\alpha)_\alpha \in J,$$

which is as close as we can come to a tuple notation for an arbitrary index set J. We denote the set of all J-tuples of elements of X by X^J.

Definition 0.8.2. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The *cartesian product* of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha,$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Definition 0.8.3. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha,$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the *box topology*.

Definition 0.8.4. Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be mapping is defined by

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta;$$

is called the *projection mapping* associated with the index β.

Definition 0.8.5. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ open in } X_\beta\},$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta.$$

The topology generated by the subbasis $\mathcal{S}$ is called the *product topology.* In this topology $\prod_{\alpha \in J} X_\alpha$ is called a *product space.*

Theorem 0.8.6. *(Comparison of the box and product topologies). The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.*

Proof. By definition of box topology, the basis for box topology on $\prod X_\alpha$ is $\mathscr{B}_b = \{\prod U_\alpha | U_\alpha \text{ is open in } X_\alpha\}$.

By definition of product topology the basis for the topology on $\prod X_\alpha$ is $\mathscr{B}_p$ then $\mathscr{B}_p$ is the collection of all finite intersection of elements of $\mathcal{S}$ where $\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$ and $\mathcal{S} = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ is open in } X_\beta\}$.

Case1:

We take finite intersection of elements of $\mathcal{S}_\beta$.

Let $\pi_\beta^{-1}(U_\beta), \pi_\beta^{-1}(V_\beta), \pi_\beta^{-1}(W_\beta) \in \mathcal{S}_\beta$.

Let $B = \pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) \cap \pi_\beta^{-1}(W_\beta)$

$=\pi_\beta^{-1}(U_\beta \cap V_\beta \cap W_\beta) \in \mathcal{S}_\beta \subset \mathscr{B}_p$

$=\pi_\beta^{-1}(U_\beta^{'})$ where $U_\beta^{'} = U_\beta \cap V_\beta \cap W_\beta$

$B = \prod_{\alpha \in J} U_\alpha^{'}$ where $U_\alpha^{'}$ is open in X_α, for $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$ and $U_\alpha^{'} = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_n$.

Case 2:

We take intersection of elements from different $\mathcal{S}_\beta$'s.

Let $B^{'} = \pi_\beta^{-1}(U_{\beta_1}) \cap \pi_\beta^{-1}(U_{\beta_2}) \cap \cdots \pi_\beta^{-1}(U_{\beta_n})$

$B^{'} = \pi_\beta^{-1}(U_{\beta_1} \cap U_{\beta_2} \cap \cdots \cap U_{\beta_n})$

Let $x = (x_\alpha)_{\alpha \in J} \in B^{'}$

Then $x = (x_\alpha)_{\alpha \in J} \in B^{'} \Leftrightarrow (x_\alpha)_{\alpha \in J} \in \pi_\beta^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_\beta^{-1}(U_{\beta_n})$

$\Leftrightarrow (x_\alpha)_{\alpha \in J} \in \cdots U_{\beta_1} \times \cdots \times U_{\beta_2} \times \cdots \times U_{\beta_n} \times \cdots$

$\Leftrightarrow x_\alpha \in U_\alpha$ for $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ and $x_\alpha \in X_\alpha$ for $\alpha \neq \beta_1, \beta_2, \cdots, \beta_n$

$\Leftrightarrow (x_\alpha) \in \prod\limits_{\alpha \in J} U_\alpha$ where U_α is open in X_α, for $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ and $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \beta_2, \cdots, \beta_n$

$B^{'} = \prod\limits_{\alpha \in J} U_\alpha$ where U_α is open in X_α.

Hence in both cases we get every basis element of the product topology in $\prod X_\alpha$ is of the form $\prod U_\alpha$ where U_α is open in X_α and $U_\alpha = X_\alpha$ except for finitely many values of α.

Clearly the basis $\mathscr{B}_p \subset \mathscr{B}_b$

Therefore, the box topology is finer than the product topology. □

Theorem 0.8.7. *Suppose the topology on each space X_α is given by a basis $\mathscr{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha,$$

where $B_\alpha \in \mathscr{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod\limits_{\alpha \in J} X_\alpha$. The collection of all sets of the same form, where $B_\alpha \in \mathscr{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod\limits_{\alpha \in J} X_\alpha$.

Proof. Let $l = \{\prod\limits_{\alpha \in J} B_\alpha \in \mathscr{B}_\alpha, \mathscr{B}_\alpha$ is a basis for $X_\alpha\}$ for each α.

B_α is a collection of open sets in X_α, for every α.

$\prod\limits_{\alpha \in J} U_\alpha$ is open in $\prod\limits_{\alpha \in J} X_\alpha$.

Therefore l is a collection of open sets in $\prod X_\alpha$.

To prove l is a basis for the box topology in $\prod_{\alpha \in J} X_\alpha$.

Now, $x = (x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$.

Let U be an open set in $\prod X_\alpha$ containing x.

Now U is an open set in the box topology in $\prod X_\alpha, x \in U$, there exists a basis element $\prod_{\alpha \in J} U_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subset U \Rightarrow x_\alpha \in U_\alpha$ for each α.

Now $x_\alpha \in U_\alpha$ and U_α is open in X_α and $\mathscr{B}_\alpha$ is a basis for X_α, there exists $B_\alpha \in \mathscr{B}_\alpha$ such that $x_\alpha \in B_\alpha \subset U_\alpha$ for each α.

Then $(x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subset \prod_{\alpha \in J} U_\alpha \subset U$.

ie, $x \in \prod_{\alpha \in J} B_\alpha \subset U$

For every $x \in \prod X_\alpha$ and any open set U containing x, there exists $\prod_{\alpha \in J} B_\alpha$ in l such that $x \in \prod_{\alpha \in J} B_\alpha \subset U$.

By 0.2.3, l is a basis for the box topology on the product space $\prod_{\alpha \in J} X_\alpha$.

Let $l' = \{\prod_{\alpha \in J} B_\alpha | \mathscr{B}_\alpha$, for finitely many indices and $B_\alpha = X_\alpha$ for the remaining indices$\}$

To prove that l' is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

Let $x = (x_\alpha) \in \prod_{\alpha \in J} X_\alpha$.

Let V be an open set in $\prod_{\alpha \in J} X_\alpha$ containing x, there exists a basis element $\prod_{\alpha \in J} U_\alpha$ for the product topology in $\prod_{\alpha \in J} X_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subset V$, where U_α is open in X_α for $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$ and $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_n$.

Now U_{α_i} is open in $X\alpha_i$ and $x_{\alpha_i} \in U_{\alpha_i}$ then there exists $B_{\alpha_i} \in \mathscr{B}_{\alpha_i}$ such that $x_{\alpha_i} \in B_{\alpha_i} \subset U_{\alpha_i}$

Define $\prod_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathscr{B}_\alpha$ for $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$.

$B_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_n$

Then clearly $\prod_{\alpha \in J} B_\alpha \in l'$ and

$x = (x_\alpha)_{\alpha \in J} \in B_\alpha \subset \prod_{\alpha \in J} U_\alpha \subset V$ for all $x \in \prod_{\alpha \in J} X_\alpha$, there exists $\prod_{\alpha \in J} B_\alpha \in l'$ such that $x \in \prod_{\alpha \in J} B_\alpha \subset V$.

By 0.2.3, l' is a basis for the product topology in $\prod X_\alpha$. □

Theorem 0.8.8. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Proof. By 0.8.7, $\prod B_\alpha$ is the basis for the subspace $\prod A_\alpha$(since $A_\alpha \subset X_\alpha$).

Therefore, $\prod A_\alpha \subset \prod X_\alpha$.

□

Theorem 0.8.9. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Proof. Write 0.8.6.

Since X_α is Hausdorff, then there are distinct neighborhoods in X_α.

Their product also containing disjoint neighborhoods.

Therefore, $\prod X_\alpha$ is Hausdorff. □

Theorem 0.8.10. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}.$$

Proof. Let $(x_\alpha) \in \prod \overline{A_\alpha}$.

To show that $(x_\alpha) \in \overline{\prod A_\alpha}$.

Let $U = \prod U_\alpha$ be a basis elements for box or product topology that contains x.

Since $x = (x_\alpha) \in \overline{A_\alpha}$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$.

Then $y = (y_\alpha) \in U$ and $\prod A_\alpha$.

Since U is arbitrary, $(x_\alpha) \in \overline{\prod A_\alpha}$.

Therefore, $\prod \overline{A_\alpha} \subseteq \overline{\prod A_\alpha}$.

Conversely, suppose $(x_\alpha) \in \overline{\prod A_\alpha}$.

To show that $(x_\alpha) \in \prod \overline{A_\alpha}$.

Let $V_\beta \in X_\beta$ containing x_β.

By definition of product topology, since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, $x_\beta \in V_\beta \subset X_\beta$.

Then $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$.

Since $A_\alpha \subset X_\alpha, y_\alpha \in \prod A_\alpha$.

Now $y_\beta \in V_\beta \cap A_\beta$

Then $x_\beta \in \overline{A_\beta}$

$\Rightarrow (x_\beta) \in \prod \overline{A_\alpha}$

$\Rightarrow \overline{\prod A_\alpha} \subseteq \prod \overline{A_\alpha}$

Therefore, $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

□

Theorem 0.8.11. *Let* $f : \prod_{\alpha \in J} X_\alpha$ *be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ *for each* α. *Let* $\prod X_\alpha$ *have the product topology. Then the fnction* f *is continuous if and only if each function* f_α *is continuous.*

Proof. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$. Let $\prod X_\alpha$ have the product topology.

Now let π_β be the projection of the product onto its βth factor.

ie, $\pi_\beta : \prod\limits_{\alpha \in J} X_\alpha \to X_\beta$.

Therefore, the function π_β is continuous.

For, if U_β is open in X_β, the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on X_α.

Now suppose $f : A \to \prod\limits_{\alpha \in J} X_\alpha$ is continuous.

Since π_β and f are continuous, the composite of these two maps, $\pi_\beta \circ f$ is continuous.

$\pi_\beta \circ f = f_\beta$ where $f_\beta : A \to X_\beta$ is continuous.

Therefore, f_β is continuous.

Conversely, suppose each function f_α is continuous.

To prove $f : A \to \prod X_\alpha$ is continuous.

$\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod X_\alpha$, where U_β is open in X_β.

$f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$

Since $f_\beta : A \to X_\beta$ is continuous, $f_\beta^{-1}(U_\beta)$ is open in A.

$f^{-1}(\pi_\beta^{-1}(U_\beta))$ is open in A.

Therefore, f is continuous. $\square$

0.9 The Quotient Topology

Definition 0.9.1. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition 0.9.2. A subset C of X is *saturated* (with respect to the surjective map $p : X \to Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the complete inverse image of a subset of Y.

Definition 0.9.3. A map $f : X \to Y$ is said to be an *open map* if for each open set U of X, the set $f(U)$ is open in Y. It is said to be a *closed map* if for each closed set A of X, the set $f(A)$ is closed in Y.

Definition 0.9.4. If X is a space and A is a set and if $p : X \to Y$ is a surjective map,then there exists exactly one topology $\mathcal{J}$ on A relative to which p is a quotient map; it is called the *quotient topology* induced by p.

Definition 0.9.5. Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element X^* containing it. In the quotient topology induced by p, the space X^* is called a *quotient space* of X.

Note: The quotient space X^* is often called an *identification space*, or a *decomposition space*, of the space X.

Theorem 0.9.6. *Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.*

(1) If A is either open or closed in X, then q is a quotient map.

(2) If p is either an open map or a closed map, then q is a quotient map.

Proof. Step 1:

First we have to prove the following two conditions:

$$q^{-1}(V) = p^{-1}(V) \text{ if } V \subset p(A);$$
$$p(U \cap A) = p(U) \cap p(A) \text{ if } U \subset X.$$

Since $V \subset p(A)$ and A is saturated, $p^{-1}(V)$ is contained in A.

Since $q : A \to p(A)$ be the map obtained by restricting $p, q^{-1}(V) \subset A$.

If $V \subset p(A)$ then $q^{-1}(V) = p^{-1}(V)$.

If $U \subset X$ also we have A be a subspace of X then we have the inclusion $p(U \cap A) \subset p(U) \cap p(A)$.

Now we have to show that $p(U) \cap p(A) \subset p(U \cap A)$.

For, suppose $y \in p(U) \cap p(A)$. Then $y = p(u) = p(a)$,for $u \in U$ and $a \in A$.

Since A is saturated, A contains every set $p^{-1}(y)$ that it intersects.

Now $A \supset p^{-1}(p(a)) \Rightarrow A \supset a$

Also $A \supset p^{-1}(p(u)) \Rightarrow A \supset u$

Then $A \supset a$ and $u \Rightarrow A \supset A \cap U$.

Since A contains every set $p^{-1}(y)$ then we get $y \in p(U \cap A)$ implies that $p(U) \cap p(A) \subset p(U \cap A)$. Therefore, $p(U \cap A) = p(U) \cap p(A)$.

Step 2:

Suppose A is open or p is open. Given the subset V of $p(A)$, we assume that $q^{-1}(V)$ is open in A. To prove that V is open in $p(A)$.

Suppose A is open. Since $q^{-1}(V)$ is open in A and A is open in X, the set $q^{-1}(V)$ is open in X.

Since $q^{-1}(V) = p^{-1}(V)$, the set $p^{-1}(V)$ is open in X. Since p is a quotient map, V is open in Y. In particular V is open in $p(A)$.

Suppose p is open. Since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, we have $p^{-1}(V) = U \cap A$, for some set U is open in X.

Now $p(p^{-1}(V)) = V$, since p is surjective; then

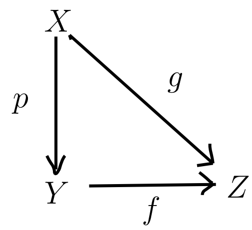
$$V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A).$$

Since p is an open map,$p(U)$ is open in Y. Hence V is open in $p(A)$.

Step 3:

When A or p is closed map then instead of "open" put "closed" in step 2. $\square$

Theorem 0.9.7. *Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.*



Proof. Suppose f is continuous. To prove g is continuous.

For each $y \in Y, p^{-1}(y)$ is open in X. Now the set $g(p^{-1}(y))$ is a one point set in Z, since g is constant on $p^{-1}(y)$.

For each $x \in X$ define a map $f : Y \to Z$ such that $f(p(x)) = g(x)$.

If f is continuous then the composite map $g = f \circ p$ is continuous. Therefore, g is continuous.

Conversely, assume g is continuous. To prove f is continuous.

Let V be open in Z, $g^{-1}(V)$ is open in X. But $g^{-1}(V) = p^{-1}(f^{-1}(V))$, since p is a quotient map.

$p^{-1}(f^{-1}(V))$ is open in X. Then $f^{-1}(V)$ is open in Y. Therefore, f is continuous.

Suppose f is a quotient map. To prove g is a quotient map.

Since g is the composite of two quotient map, g is a quotient map.

Conversely, assume g is a quotient map. Since g is surjective and so f is surjective.

Let V be a subset of Z. Now $f^{-1}(V)$ is open in Y.

Since p is continuous, $p^{-1}(f^{-1}(V))$ is open in X.

We have $p^{-1}(f^{-1}(V)) = g^{-1}(V), g^{-1}(V)$ is open in X. Then V is open in Z.

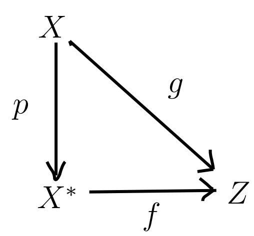
Therefore, f is a quotient map. □

Corollary 0.9.8. *Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:*

$$X^* = \{g^{-1}(\{z\})|z \in Z\}.$$

Give X^* the quotient topology.

(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.



(b) If Z is Hausdorff, so is X^*.

Proof. (a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism then both f and the projection map $p : X \to X^*$ are quotient map. ie, $g = f \circ p$ is a quotient map.

Conversely, suppose g is a quotient map. By 0.9.7, f is a quotient map. since f is bijective, f is a homeomorphism.

(b) Suppose Z is Hausdorff. Then U and V are disjoint neighborhood under f. Since $f : X^* \to Z$ is a homeomorphism. Then $f^{-1}(U)$ and $f^{-1}(V)$ are the disjoint neighborhood under X^*. Therefore, X^* is a Hausdorff. $\square$

Unit 3

0.10 Connected spaces

Definition 0.10.1. *Let X be a topological space. A separation of X is a pair (u, v) of disjoint non empty open subsets of X whose union is X.*

Definition 0.10.2. *The space X is said to be connected if there dose not exists a separation of X.*

Remark 0.10.3. *If X is connected, then any space homomorphic to X is connected.*

Theorem 0.10.4. *A space X is connected iff the only subsets of X that are both open and closed are the empty set and X itself.*

Proof. First assume X is connected.

Claim : The only subsets of X that are both open and closed are the empty set and X itself.

For, suppose A is a nonempty proper subset of X. That is both open and closed in X.

We have $X - A$ is nonempty. If we take A is closed in X. Then $X - A$ is open.

Therefore we have two nonempty disjoint open sets A and $X - A$ such that their union is X.

That is A and $X - A$ forms a separation of X.

$\Rightarrow X$ is not conncted.

This contradication asserts our claim.

Conversely, assume the only subsets of X that are both open and closed are empty and X itself.

Claim : X is connected.

For, if X is not connected, there is a separation of X.

Let U and V forms the separation. Therefore U is nonempty.

U is open $\Rightarrow X - U$ is closed in X.

$\Rightarrow V$ is closed in X.

Also, V is open $\Rightarrow X - V$ is closed in X.

$\Rightarrow U$ is closed in X.

Thus we have U is a proper subset of X. That is both open and closed. This is a contradication.

Therefore X is connected. □

Lemma 0.10.5. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y connected if there exists no separation of Y.*

Proof. Let Y be a subspace of X.

To prove separation of Y iff A and B are two nonempty disjoint sets such that $A \cup B = Y$, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

First assume that there exists a separation of Y . Then there exists disjoint nonempty open subsets A and B such that $A \cup B = Y$.

It is enough to prove $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

Then A is both open and closed in Y.

The closure of A in Y is $\overline{A} \cap Y$ where $\overline{A}$ denote the closure of A in Y.

Since Anis closed in X. $A = \overline{A} \cap Y$ where $\overline{A}$ is the closure of A in X. To say the same thing $\overline{A} \cap B = \emptyset$. Since $\overline{A}$ is the union of A and its limit points, B contains no limit points of A.

Similarly, we can show that A conatins no limit points of B.

Conversely, assume A and B are two nonempty disjoint sets such that $A \cap B = Y$, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Claim :$\overline{A} \cap Y = A$.

We have A is contained $\overline{A}$ and $A \subset Y$.

That is $A \subset \overline{A}$ and $A \subset Y$.

Therefore $A \subset \overline{A} \subset Y$————————(1)

Now, let $x \in \overline{A} \subset Y$. Then $x \in \overline{A}$ and $x \in Y$.

Therefore, $x \notin B$ and $x \in Y$.

$\Rightarrow x \in A$ $(since\ Y = A \cup B)$.

Therefore, $\overline{A} \cap Y \subset A$ ————————-(2).

From (1) and (2) we get, $A = \overline{A} \cap Y$.

Similarlly, we can prove $\overline{B} \cap Y = B$.

Now, $\overline{A}$ is closed in X.

$\Rightarrow$ $\overline{A} \cap Y$ is closed in Y.

$\Rightarrow$ A is closed in Y.

Similarlly, B is closed in Y.

Now, $B = Y - A$ is open in Y.

Therefore, B is open in Y.

Also $A = Y - B$.

Therefore, A is open in Y.

Thus A and B are two nonempty disjoint open sets in Y with $Y = A \cup B$.

Thus there exists a separation of Y. □

Lemma 0.10.6. *If the sets C and D form a separation of X and if Y is connected* *subspace of X, then Y lies entirely with in either C or D.*

Proof. Let sets C and D form a separation of X.

Therefore, $X = C \cup D$ where C and D are nonempty disjoint open sets in X.

Let Y be a connceted subspace of X.

To prove Y lies entirely with in either C or D.

Since C and D are open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y.

Also, $Y = Y \cap X$

$$= Y \cap (C \cup D)$$

$$= (Y \cap C) \cup (Y \cap D).$$

Now, $(Y \cap C) \cap (Y \cap D) = Y \cap (C \cap D)$

$$= Y \cap \emptyset$$

$$= \emptyset$$

Therefore, these two sets are disjoint and their union is Y.

If $C \cap Y$ and $D \cap Y$ are both nonempty.

Then they would constitute a separation of Y. Since Y is connceted, the only posibility is $Y \cap C = \emptyset$ or $Y \cap D = \emptyset$. Therefore, $Y \subset C$ or $Y \subset D$. Thatis, Y is entirely either in C or in D. □

Example 0.10.7. *Let X denote a two points space in the indiscrete topology. Obviously there is no separation of X, so X is connected.*

Example 0.10.8. *Let Y denote the subspace $[-1, 0) \cup (0, 1]$ of the real line R each of the sets $[-1, 0)$ and $(0, 1]$ is nonempty and open in Y. They form a separation of Y.*

Example 0.10.9. *Let X be the subspace $[-1, 1]$ of the real line. The sets $[-1, 0)$ and $(0, 1]$ are disjoint and nonempty, but they does not form the separation of X.*

Because the first set is not open in X.

Example 0.10.10. *The rationals Q are not connected.*

Lemma 0.10.11. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of connected subspaces of X that have a common point. Let $p \in A_\alpha$ for each α be the common point. To prove $\bigcup A_\alpha$ is connected. Let $Y = \bigcup A_\alpha$.

Suppose Y is not connected. Then there is a separation of Y. That is there exixt C and D are two nonempty disjoint open sets in Y such that $C \cup D = Y$.

We have $p \in Y$, therefore $p \in C$ or $p \in D$.

For, definteness let $p \in C$

Therefore, we have $p \in A_\alpha$

$\Rightarrow A_\alpha \subset C$ for each α

$\Rightarrow \bigcup A_\alpha \subset C$

That is $Y \subset C$

$\Rightarrow D$ is empty.

This is a contradication to D is nonempty. Therefore, Y is connceted. Thae is $\bigcup A_\alpha$ is connected. □

Theorem 0.10.12. *Let A be a connected subspace of X and if $A \subset B \subset \overline{A}$. Then B is also connected.*

Proof. Let A be a connected subspace of X and let $A \subset B \subset \overline{A}$.

To prove B is connceted.

Suppose B is not connected. Then we can write, $B = C \cup D$ where C and D are nonempty set with $\overline{C} \cap D = C \cap \overline{D} = \emptyset$.

We have, $A \subset B$

$\Rightarrow A \subset C \cup D$.

Since A is connceted, By a theorem, $A \subset C$ or $A \subset D$.

Assume that, $A \subset C$

$\Rightarrow \overline{A} \subset \overline{C}$

$\Rightarrow B \subset \overline{C}$

$\Rightarrow B \cap D = \emptyset$.

But $B = C \cup D$. Therefore, $D = \emptyset$.

Which is a contradication to D is a nonempty set. Therefore, our assumtion is wrong. Therefore, B is connected. □

Theorem 0.10.13. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \to Y$ be a continuous map. Given X is connected.

To prove f(X) is connected.

Suppose f(X) is not connected. Then we can write, $f(X) = A \cup B$ where A and B are nonempty disjoint open set in f(x).

Let $g : X \to f(X)$ with g(x) = f(x), $\forall x \in X$. Then g is onto and continuous.

Now, $X = g^{-1}(f(x))$

$= g^{-1}(A \cup B)$

$= g^{-1}(A) \cup g^{-1}(B)$.

Since g is continuous and A and B are nonempty open set in $g^{-1}(A)$ and $g^{-1}(B)$ are open. Therefore,$g^{-1}(A)$ and $g^{-1}(B)$ are open in X.

Thus $X = g^{-1}(A) \cup g^{-1}(B)$ where $g^{-1}(A)$ and $g^{-1}(B)$ are nonempty open set with $g^{-1}(A) \cap g^{-1}(B) = \emptyset$.

Therefore, X is not connected.

Which is a contradication to X is connected. Therefore, our assumption is wrong.

Therefore, f(x) is connceted. □

Theorem 0.10.14. *A finite cartesian product of connected space is connected.*

Proof. Let $X_1, X_2, \ldots, X_n$ be connected spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is connected.

First we prove product of two connected spaces X is connected.

Choose a base point $a \times b$ in the product $X \times Y$. Note that, the horizontal slice $X \times b$ is connected being homeomorphic with X and each vertical slice $X \times Y$ is connected being homeomorphic with Y.

For each $x \in X$, define T-shaped space, $T_x = (X \times b) \cup (x \times Y)$.

We have $x \times b \in X \times b$ and $x \times b \in x \times Y$.

Therefore, $x \times b \in (x \times b) \cap (x \times Y)$.

$\Rightarrow (x \times b) \cap (x \times Y) \neq \emptyset$.

By a theorem, $x \times b \cup x \times Y$ is connected. Therefore, T_x is connected for every $x \in X$.

Claim : $X \times Y = \bigcup_x T_x$

Clearly, $T_x \subseteq X \times Y$ for every $x \in X$.

Therefore, $\bigcup_{x \in X} T_x \subseteq X \times Y$ ——————(1).

Now, To prove $X \times Y \subseteq \bigcup_{x \in X} T_x$.

We have, $x \times y \in X \times Y$

$x \times Y \in x \times Y \subset T_x$

$x \times y \in T_x \subseteq \bigcup T_x$

$X \times Y \subseteq \bigcup_{x \in X} T_x$ ——————-(2).

From equations (1) and (2) we get, $X \times Y = \bigcup_{x \in X} T_x$.

We have $(a,b) \in X \times b$

Therefore, $(a,b) \in T_x \ \forall\ x \in X$.

Therefore, $\bigcap_{x\in X} T_x \neq \emptyset$.

Thus $X \times Y = \bigcup_{x\in X} T_x$ where $\bigcap_{x\in X} T_x \neq \emptyset$.

By a lemma, $X \times Y$ is connected as each T_x is connected.

Now, we prove that cross product of finite number of connected spaces is connected.

Let $X_1, X_2, \ldots, X_n$ be n-connected spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is connected.

By the observation, we say that $X_1 \times X_2$ is connected. Therefore, the result is true for $n = 2$.

Assume that the result is true for n-1.

That is $X_1 \times X_2 \times \ldots \times X_{n-1}$ is connected.

To prove the result is true for n.

We have, $X_1 \times X_2 \times \ldots \times X_n$ is homeomorphic with $(X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$.

By our assumption, $(X_1 \times X_2 \times \ldots \times X_{n-1})$ is connected. Therefore,$(X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$ is connected.

$\Rightarrow (X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$ is connected. □

0.11 Compact spaces

Definition 0.11.1. *A collection $\mathscr{A}$ of subsets of X is said to be cover X or to be a covering of X if the union of elements of $\mathscr{A}$ is equal to X.*

Definition 0.11.2. *A collection $\mathscr{A}$ of open subsets of X is said to be a open covering of X if its union of elements of $\mathscr{A}$ is equal to X.*

Definition 0.11.3. *A space X is said to be compact if every open covering $\mathscr{A}$ of X contains a subcollection that also covers X.*

Example 0.11.4. *The real line R is not connected.*

Let $\mathscr{A} = \{(n, n+2)/n \in Z\}$ be a collection of open subsets of R whose union is R. But this collection does not have a finite subcollection that covers R.

Example 0.11.5. *Let $X = \{0\} \cup \{\frac{1}{n}/n \in Z_+\}$ be a subspace of R. Then X is compact. Let $\{U_\alpha\}$ be an open covering of X. Therefore, $X = \bigcup_\alpha U_\alpha$.*

$0 \in X \Rightarrow 0 \in \bigcup_\alpha U_\alpha$

$\Rightarrow 0 \in U_\alpha$ *for some* α.

U_α is an open set containing zero. Therefore, U_α is a neighbourhood of zero. Since $\frac{1}{n} \to 0$, there exists a positive integer N such that $\frac{1}{n} \in U_\alpha\ \forall\ n \geq N$.

$\Rightarrow \frac{1}{N}, \frac{1}{N+1}, \ldots, 0 \in U_\alpha$.

Now, $1, \frac{1}{2}, \ldots, \frac{1}{N-1}$ are in $\bigcup U_\alpha$.

Let $1 \in U_{\alpha_1}, \frac{1}{2} \in U_{\alpha_2}, \ldots, \frac{1}{N-1} \in U_{\alpha_{N-1}}$.

Therefore, $\{1, \frac{1}{2}, \ldots, \frac{1}{N-1}, \frac{1}{N}, \frac{1}{N+1}, \ldots, 0\} \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_{N-1}} \cup U_\alpha$

$\Rightarrow X \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_{N-1}} \cup U_\alpha$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_{N-1}}, U_\alpha\}$ is a finite subcollection which covers X. Therefore, X is compact.

Example 0.11.6. $(0, 1]$ *is not compact. Since the open covering $\mathscr{A} = \{(\frac{1}{n}, 1)/n \in Z_+\}$ contains no finite subcollection covering* $(0, 1]$

Example 0.11.7. $(0, 1]$ *is not compact and* $[0, 1]$ *is compact.*

Definition 0.11.8. *If Y is the subspace of X, a collection $\mathscr{A}$ of subset of X is said to cover Y if the union of this element contains Y.*

Lemma 0.11.9. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y bysets open in X contains a finite subcollection covering Y.*

Proof. First assume Y is compact and let $\mathscr{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y bysets open in X.

Now, consider the collection $\{A_\alpha \cap Y\}_{\alpha \in J}$ this is the covering of Y bysets open in Y.

Since $A_\alpha \cap Y$ is open in Y for each α. Therefore, by compactness of Y, this collection has a finite subcollection $\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, A_{\alpha_3} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$ that covers Y.

Then $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ is the finite subcollection of A that covers Y.

Conversely, assume every covering of Y bysets open in X contains a finite subcollection covering Y.

To prove Y is compact.

Let $A' = \{A'_\alpha\}$ be a covering of Y bysets open in X.

For, each α choose a set A_α open in X such that $A'_\alpha = A_\alpha \cap Y$.

$Y = A'_{\alpha_1} \cup A'_{\alpha_2} \cup \ldots \cup A'_{\alpha_i} \cup \ldots$

$Y = (A_{\alpha_1} \cap Y) \cup (A_{\alpha_2} \cap Y) \cup \ldots \cup (A_{\alpha_i} \cap Y) \ldots$

$\quad = Y \cap (A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots)$

$Y \subset A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_i} \cup \ldots.$

The collection $\{A_\alpha\}$ is the covering of Y bysets open in X. Therefore, by hypothesis, some finite subcollection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ covers Y.

Then $\{A'_{\alpha_1}, A'_{\alpha_2}, \ldots, A'_{\alpha_n}\}$ is the subcollection of A' that covers A. Therefore, Y is compact. □

Theorem 0.11.10. *Every closed subsets of a cmpact space is compact.*

Proof. Given X is compact. Let Y be a closed subset of a compact set X.

To prove Y is compact.

Let $A = \{A_\alpha\}_{\alpha \in J}$ be a covering of Y bysetsopen in X.

Let us form an open covering β of Y by adjoining to A, single open set X-Y.

Since X is compact, there exists a finite subcollection $\{A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_n} \cup X - Y\}$ of β that covers X. Therefore, $X = \{A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_n} \cup X - Y\}$.

Then $Y \subset A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_n}$.

$\Rightarrow$ There exists a finite subcollection of A which covers Y. Therefore, by previous lemma, Y is compact. □

Theorem 0.11.11. *Every compact subset of a hausdorff space is closed.*

Proof. Let X be a hausdorff space. Let Y be a compact space of X.

To prove Y is closed in X.

That is to prove X-Y is open in X.

Let $x_0 \in X - Y$

$\Rightarrow x_0 \notin Y$

Then $x_0 \neq y \; \forall \; y \in Y$.

Now, x_0 and y are two distinct points of Hausdorff space X.

For, each point y of Y, there exists a disjoint neighbourhood U_y and V_y of x_0 and y respectively.

Now, the collection $\{V_y / y \in Y\}$ is the collection of open in X and $Y \subset \bigcup_{y \in Y} V_y$.

Therefore, $\{V_y / y \in Y\}$ is the covering of Y bysets open in X.

By lemma, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers Y.

That is $Y \subset V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$.

Let $V = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$. Then $Y \subset V$ and V is open in X.

Let $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}$.

Therefore, U is the finite intersection of open sets containing x_0.

Therefore, U is an open sets containing x_0.

Claim: $U \cap V = \emptyset$.

Suppose $U \cap V \neq \emptyset$. Then $z \in U \cap V$

$\Rightarrow z \in U \text{ and } z \in V$.

Now, $z \in U \Rightarrow z \in U_{y_i}\ \forall\ i = 1, 2, \ldots, n$.

Also $z \in V \Rightarrow z \in V_{y_i}$ for some i.

$z \in U_{y_i} \cap V_{y_i}$.

Which is a contradication to $U_{y_i} \cap V_{y_i} = \emptyset$.

Therefore, $U \cap V = \emptyset$. Also $Y \subset U$.

$\Rightarrow U \cap Y = \emptyset$

$\Rightarrow U \subset X - Y$

$\Rightarrow X - Y$ is open in X.

$\Rightarrow$ Y is closed in X. □

Theorem 0.11.12. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be a continuous map, where X is a compact space and Y be a topological space.

To prove $f(X)$ is compact.

Let $\mathscr{A}$ be a cover of $f(X)$ bysets open in Y. Then $f(X) \subset \bigcup_{A \in \mathscr{A}} A$. Since f is continuous and A is open in Y.

$\Rightarrow f^{-1}(A)$ is open in X for every $A \in \mathscr{A}$.

Also, $X = \bigcup_{A \in \mathscr{A}} f^{-1}(A)$.

Therefore, $\{f^{-1}(A)/A \in \mathscr{A}\}$ is an open covering for X.

Since X is compact, there exists a finite subcollection, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$ that covers X.

That is $X = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \ldots \cup f^{-1}(A_n)$

$\Rightarrow f(X) \subset A_1 \cup A_2 \cup \ldots \cup A_n$.

$\{A_1, A_2, \ldots, A_n\}$ is a finite subcollection of $\mathscr{A}$ that covers f(X).

By a lemma, $f(X)$ is compact. $\square$

Theorem 0.11.13. *Let $f : X \to Y$ be a bijective continuous function, if X is compact and Y is hausdorff, then f is a homeomorphim.*

Proof. Let $f : X \to Y$ be a bijective continuous function. Given X is compact and Y is hausdorff.

To prove f is a homeomorphic.

It is enough to prove f^{-1} is continuous.

That is to prove that $(f^{-1})^{-1}(A)$ is closed in Y, for every closed set A in X.

Thatis, to prove f(A) is closed in Y for every closed set A in X.

Let $A \subset X$ be closed in X.

Now, A being closed subset of the compact set X, A is compact.

Now, $f(A)$ being a continuous image of a compact set A, $f(A)$ is compact.

Again, $f(A)$ being a compact subset of a hausdorff space Y.

Therefore, $f(A)$ is closed.

Therefore, f^{-1} is continuous.

Therefore, f is a homeomorphism. $\square$

Theorem 0.11.14. *The product of finitely many compact space is compact.*

Proof. Let $X_1, X_2, \ldots, X_n$ be compact spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is compact.

First we shall prove that the product of two compact space is compact.

Then the theorem follows by induction for any finite product.

Before proving this theorem, let us prove the **Tube lemma**. Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times y$ where W is a neigbourhood of $x_0$in X.

We prove the following, there is a neighbourhood W of x_0 in X such that $W \times Y \subset N$.

$W \times Y$ is often called a tube about $x_0 \times Y$.

First let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$ lieing in N).

The space $x_0 \times Y$ is compact being homeomorphic to Y.

We can cover $x_0 \times Y$ by finitely many such basis element $U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n$.

We assume that each basis element $U_i \times V_i$ intersects $x_0 \times Y$.

Since otherwise the basis element would be super fluous we can discard it forms the finite collection and still the covering of $x_0 \times Y$.

Define $W = U_1 \cap U_2 \cap \ldots \cap U_n$.

Then the set W is open and it contains x_0 because each $U_i \times V_i$ intersects $x_0 \times Y$.

we assume that the sets $U_i \times V_i$ which were choose to cover $x_0 \times Y$ actually cover the tube $W \times Y$.

For, let $X \times y \in W \times Y$.

Consider the point $x_0 \times y$ of the slice $x_0 \times Y$, having the same y-coordinate at this point.

Now, $x_0 \times y \in U_i \times V_i$ for some i.

So that $y \in V_i$.

But $x \in U_j$ for all j.

We have $x \times y \in U_i \times V_i$. Therefore, $W \times Y \subset N$. Hence the lemma.

Proof of the main theorem:

Let X and Y be compact space.

To prove $X \times Y$ is compact.

Let $\mathscr{A}$ be an open covering of $X \times Y$.

Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and therefore it can be covered by finitely many elements $A_1, A_2, \ldots, A_m$ of $\mathscr{A}$.

Their union $N = A_1 \cup A_2 \cup \ldots \cup A_m$ is an open set containing $x_0 \times Y$.

By above tube lemma, the open set N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X.

Then $W \times Y$ is covered by finitely many elements $A_1, A_2, \ldots, A_m$ of $\mathscr{A}$.

Thus for each $x \in X$, we can choose a neigbourhood W_x of X such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathscr{A}$.

Since X is compact. There exists a finite subcollection $\{W_1, W_2, \ldots, W_k\}$ which covers X.

Therefore, theunion of the tubes $W_1 \times Y, W_2 \times Y, \ldots, W_k \times Y$ covers all of $X \times Y$.

Since each may be covered by finitely many elements of $\mathscr{A}$.

Hence $X \times Y$ has a finite subcover. Thus $X \times Y$ is compact.

By induction, it follows that $X_1, X_2, \ldots, X_n$ are compact spaces then their product $X_1 \times X_2 \times \ldots \times X_n$ is compact.

Definition 0.11.15. *A collection $\mathscr{C}$ of subsets of X is said to satisfy the finite intersection properly if for every finite subclooection $\{C_1, C_2, \ldots, C_n\}$ of $\mathscr{C}$, the intersection $C_1 \cap C_2 \cap \ldots \cap C_n$ is nonempty.*

Theorem 0.11.16. *Let X be a tropological space. Then X is compact if and only if for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{c \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is nonempty.*

Proof. Suppose X is compact.

Let $\mathscr{C}$ be a collection of closed sets in X satisfiying the finite intersection condition.

To prove $\bigcap_{C\in\mathscr{C}} C \neq \emptyset$.

If not assume,$\bigcap_{c\in\mathscr{C}} C = \emptyset$.

Then $X - \bigcap_{c\in\mathscr{C}} C = X - \emptyset$.

Since C is closed for all $C \in \mathscr{C}$, X- C is open for all $C \in \mathscr{C}$. Therefore,$\{X - C/c \in \mathscr{C}\}$ is a collection of open subsets of X and $X = \bigcap_{C\in\mathscr{C}}(X - C)$.

Therefore, $\{X - C/C \in \}$ is an open cover for X. Since X is compact, there exists a finite subcollection, $\{X - C_1, X - C_2, \ldots, X - C_n\}$ which covers X.

Therefore, $X = (X - C_1) \cup (X - C_2) \cup \ldots \cup (X - C_n)$

$\Rightarrow X = X - (C_1 \cap C_2 \cap \ldots \cap C_n)$

$\Rightarrow C_1 \cap C_2 \cap \ldots \cap C_n = \emptyset$.

Which is a contradication to $\mathscr{C}$ satisfies the finite intersection condition, $\bigcap_{C\in\mathscr{C}} C \neq \emptyset$.

Conversely, suppose that for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C\in\mathscr{C}} C$ of all elements of $\mathscr{C}$ is nonempty.

To prove X is compact.

Suppose X is not compact.

Then there exists an open covering $\mathscr{A}$ for X which contains no finite subcovering.

Since $\mathscr{A}$ is an open covering for X.

$X = \bigcup_{A\in\mathscr{A}} A$. Then $X - X = X - \bigcup_{A\in\mathscr{A}} A$.

That is $\emptyset = \bigcap_{A\in\mathscr{A}}(X - A)$ ————-(1)

Now, $\{X - A/A \in \mathscr{A}\}$ is a collection of closed sets in X.

Let $\{X - A_1, X - A_2, \ldots, X - A_n\}$ be a subcollection of $\{X - A/A \in \mathscr{A}\}$.

Then $(X - A_1) \cap (X - A_2) \cap \ldots \cap (X - A_n) = X - (A_1 \cup A_2 \cup \ldots \cup A_n) \neq \emptyset$.

Therefore, $\{X - A/A \in \mathscr{A}\}$ is a collection of closed subsets of X satisfying the finite intersection condition and by (1) $\bigcap_{A\in\mathscr{A}}(X - A) = \emptyset$.

Which is a contradication.

Therefore, our assumption is wrong.

Hence X is compact. □

Unit 4

0.12 The countability Axioms

Definition 0.12.1. *A space X is said to have a countable basis at x if there is a countable collection $\mathscr{B}$ of neighbourhood of x such that each neihbourhood of x contains at least one of the elements of $\mathscr{B}$.*

Definition 0.12.2. *A space that has a countable basis at each of its points is said to satisfy the first countability axiom or to be first countable.*

Theorem 0.12.3. *Let X be a topological space. (a) Let A be a subset of X. If there is a sequence of points of A coverging to x then $x \in \overline{A}$; the converse holds if X is first countable. (b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$ the converse holds if X is first countable.*

Proof. (a) Suppose $x \in \overline{A}$. Since X is first countable, there exists a countable basis say U_n at x.

Let $A_n = U_1 \cap U_2 \cap \ldots \cap U_n$ for $n = 1, 2, \ldots$

Then $\{A_n\}$ is a countable collection of neighbourhood of x and $A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots$

Claim : $\{A_n\}$ is a countable basis at x.

Let U be a neihbourhood of x. Since U_n is a countable basis at x, there exists U_k in $\{U_n\}$ such that $U_k \subset U$.

Also, $A_k \subset U_k$. Therefore, we have $A_k \subset U_k \subset U$.

That is $x \in A_k \subset U$.

Therefore, $\{A_n\}$ is a countable basis at x.

Now, for any n, $A_n \cap A \neq \emptyset$.

Choose $x_n \in A_n \cap A$ for $n = 1, 2, \ldots$

Now, we have a sequence (x_n) in A such that $x_n \in A_n$ for $n = 1, 2, \ldots$.

Claim : $(x_n) \to x$.

Let V be a neigbourhood of x.

Since $\{A_n\}$ is a countable basis at x, there exists x such that $A_N \subset V$.

Also, $A_n \subset A_N \; \forall \; n \geq N$.

Therefore, $x_n \in A_n \subset V$

$\Rightarrow x_n \in V \; \forall \; n \geq N$.

Therefore, $(x_n) \to x$.

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \to x$.

To prove $x \in \overline{A}$

Suppose there exists a sequence of points in A converging to x.

Let W be a neighbourhood of x.

Since $(x_n) \to x$ and W is a neighbourhood of x, there exists a positive integer N such that $x_n \in W, \; \forall \; n \geq N$

$\Rightarrow W \cap A \neq \emptyset$.

Therefore, $x \in \overline{A}$.

Suppose $f : X \to Y$ is continuous.

To prove $(f(x_n)) \to f(x)$ where $(x_n) \to x$.

Let $(x_n) \to x$. Let V be the neigbourhood of f(x).

$\Rightarrow f^{-1}(V)$ is the neigbourhood of x.

Since $(x_n) \to x$, there exists a positive integer N such that $x_n \in f^{-1}(V), \; \forall \; n \geq N$

$\Rightarrow f(x_n) \in V \; \forall \; n \geq N$.

Therefore, $(f(x_n)) \to f(x)$.

Conversely, suppose that $(f(x_n)) \to f(x)$ whenever $(x_n) \to x$.

To prove f is continuous.

It is enough to prove $f(\overline{A}) \subset \overline{f(A)}$ for any subset A of X.

Let $y \in f(\overline{A})$. Then $y = f(x)$ for some $x \in \overline{A}$.

Now, $x \in \overline{A}$. By (a), there exists a sequence (x_n) in A such that $(x_n) \to x$.

By hypothesis, $(f(x_n)) \to f(x)$.

Then by (a), $f(x) \in \overline{f(A)}$

$\Rightarrow y \in \overline{f(A)}$.

Therefore, $f(\overline{A}) \subset \overline{f(A)}$.

Hence f is continuous. □

Example 0.12.4. *1. $\mathbb{R}$ has a countable basis. It is the collection of all open intervals (a, b) with rational end points.*

2. $\mathbb{R}^n$ has a countable basis. It is the collection of all products of intervals having rational end points.

3. $\mathbb{R}^w$ has a countable basis. It is the collection of all product $\Pi_{n \in Z_+} U_n$ where U_n is an open interval with rational end points for finitely many values of n and $U_n = \mathbb{R}$ for all values of n.

Definition 0.12.5. *If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom or to be second countable.*

Theorem 0.12.6. *(i) A subspace of a first countable space is first countable and a countable product of first countable spaces is first countable. (ii) A subspace of a second countable space is second countable and a countable product of second countable space is second countable.*

Proof. (i) Let A be a subspace of a first countable space X.

Let $x \in X$.

Let $\mathscr{B}$ be a countable basis for X.

Let $\mathscr{C} = \{B \cap A / B \in \mathscr{B}\}$.

Then $\mathscr{C}$ is a countable basis for the subspace A of X. Therefore, A is first countable.

Let (X_i) be a sequence of first countable spaces.

To prove ΠX_i is first countable.

Let $\mathscr{B}_i$ be a countable basis for the space X_i.

Then the collection of all products ΠU_i where $U_i \in \mathscr{B}_i$ for finitely many values of i is a countable basis for ΠX_i. Therefore, ΠX_i is first countable.

(ii) Consider the second countability axiom. Let X be a second countable space.

Let A be a subspace of X.

Let $\mathscr{B}$ be a countable basis for X.

Let $\mathscr{C} = \{B \cap A / B \in \mathscr{B}\}$.

Then $\mathscr{C}$ is a countable basis for the subspace A of X. Therefore, A is second countable.

Therefore, any subspace of a second countable space is second countable.

Let (X_i) be a sequence of second countable spaces.

To prove ΠX_i is second countable.

Let $\mathscr{B}_i$ be a countable basis for the space X_i.

Then the collection of all products ΠU_i where $U_i \in \mathscr{B}_i$ for finitely many values of i is a countable basis for ΠX_i. Therefore, ΠX_i is second countable. □

Definition 0.12.7. *A space A of a space X is said to be dense in X if* $\overline{A} = X$

Theorem 0.12.8. *Suppose that X has a countable basis. Then (a) Every open covering of X contains a countable subcollection covering X.*
(b) There exists a countable subset of X. That is, dense in X.

Proof. Given X as a countable basis.
Let $\{B_n\}$ be a countable basis for the topology on X.

(a) Let $\mathscr{A}$ be an open covering for X.
For each positive integer n for which it is possible to choose an element A_n of $\mathscr{A}$ containing the basis element B_n.
That is $B_n \subset A_n$
Let $\mathscr{A}' = \{A_n\}$, then clearly $\mathscr{A}'$ is the countable collection of open subsets of X.
To prove $X = \bigcup A_n$. Trivially, $\bigcup A_n \subset X$ ————-(1)
Let $x \in X$
$\Rightarrow x \in A$ for some $A \in \mathscr{A}$.
There exists $B_n \in \{B_n\}$ such that $x \in B_n \subset A$.
Since $B_n \subset A_n$
$\Rightarrow x \in \bigcup A_n$.
Therefore, $X \subset \bigcup A_n$ ————-(2).
From (1) and (2) we get, $X = \bigcup A_n$.
Therefore, $\mathscr{A}'$ is a countable subcollection covering X.

(b) For each nonempty basis element B_n, choose a point $x_n \in B_n$.
Let D be the set consisting of the point x_n.
Clearly, D is the countable subset of X.
Claim : $\overline{D} = X$
Clearly, $\overline{D} \subset X$.
To prove $X \subset \overline{D}$.

Let $x \in X$.

Let U be a neihbourhood of x.

Then there exists B_n such that $x \in B_n \subset U$.

Now, $x_n \in B_n,\ x_n \in D$

$\Rightarrow x_n \in B_n \cap D$

$\Rightarrow B_n \cap D \neq \emptyset$

$\Rightarrow x \in \overline{D}$.

Therefore, $x \subset \overline{D}$. Hence $\overline{D} = X$.

Therefore, D is dense in X. □

Definition 0.12.9. *A space for which every open covering contains a countable subcovering is called a Lindelof space.*

Definition 0.12.10. *A space having a countable dense subset often said to be separable.*

Example 0.12.11. *The space $\mathbb{R}_l$ satisfies all the countability axioms but the seconds or $\mathbb{R}_l$ topology is first countable but not second countable.*

Proof. Let $x \in \mathbb{R}_l$, the set of all elements of the form $[x, x + \frac{1}{n})$ is a countable basis at x and it is easy to see that the rational number of dense in $\mathbb{R}_l$. Hence it is first countable.

To show $\mathbb{R}_l$ is not second countable.

Let $\mathscr{B}$ be a basis for $\mathbb{R}_l$.

Choose for each x, an element B_i of $\mathscr{B}$ such that $x \in B_x \subset [x, x + 1)$.

If $x \neq y$, then $B_x \neq B_y$.

Since $x = \inf B_x$ and $y = \inf B_y$.

Therefore, $\mathscr{B}$ must be countable.

Therefore, it does not satisfy the second countability axiom. □

Example 0.12.12. *The product of two Lindelof spaces need not be Lindelof.*

(or)

$\mathbb{R}_l$ *is Lindelof but the product* $\mathbb{R}_l \times \mathbb{R}_l$ *is not Lindelof.*

Proof. The space $\mathbb{R}_l^2$ has basis of all sets of the form $[a, b) \times [c, d)$.

We show that it is not Lindelof.

Consider a subspace $L = \{x \times (-x)/x \in \mathbb{R}_l\}$ and L is closed in $\mathbb{R}_l^2$.

Let us cover $\mathbb{R}_l^2$ by the open set $\mathbb{R}_l^2 - L$ and by all elements of the form $[a, b) \times [-a, d)$.

Each of these open sets intersects L in atmost one point.

Since L is uncountable, no countable subcollection covers $\mathbb{R}_l^2$.

Therefore, $\mathbb{R}_l^2$ is not Lindelof.

The subspace of a Lindelof space need not be Lindelof.

The ordered square, I_0^2 is compact.

Therefore, it has a countable subcover.

Therefore, it is Lindelof trivially.

Now, consider the subspace $A = I \times (0, 1)$ of I_0^2.

It is not Lindelof.

For, A is the union of disjoint sets, $U_x = \{x\} \times (0, 1),\ x \in I$ each of which is open in A.

This collection of sets is uncountable and no proper subcollection covers A.

It is not Lindelof. □

Note: $\mathbb{R}_l^2$ is called sorgenfrey plane.

Definition 0.12.13. *Suppose that one point sets are closed in X. Then X is said to be* ***regular*** *if for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B respectively.*

Definition 0.12.14. *Suppose that the one point sets are closed in X. Then X is said to be* ***normal*** *if for each pair (A, B) of disjoint closed sets of X, there exists disjoint open sets containing A and B respectively.*

Note: A regular space is hausdroff and normal space is regular.

Lemma 0.12.15. *Let X be a topological space. Let one point in X be closed.*
(a) X is regular if and only if given a point x of X and a neigbourhood U of x, there is a neihbourhood V of x such that $\overline{V} \subset U$.
(b) X is normal if and only if given a closed set A of an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. (a) First assume X is regular.
Given a point x and a neigbourhood U of x.
To prove there exists a neigbourhood V of x such that $\overline{V} \subset U$.
Let $B = X - U$.
Then B is closed in X.
Also $x \notin B$.
Therefore, by hypothesis, there exists disjoint open sets V and W containing x and B respectively.
Therefore, the set $\overline{V}$ is disjoint from B.
Since if $y \in B$ the set W is a neigbourhood of x such that $\overline{V} \subset U$.
To prove X is regular.
Suppose the closed set B not containing x be given. Then $x \in U$.
By hypothesis, there is a neighbourhood V of x such that $\overline{V} \subset U$.
Therefore, the open sets V and $X - \overline{V}$ are disjoint open set containing x and B respectively.
Hence X is regular.

(b) Suppose that X is normal.

Given a closed set A and an open set U containing A.

Let $B = X - U$.

Since U is open, B is closed in X.

Also we have A is closed in X.

Since X is normal, there exist disjoint open sets V and W containing A and b respectively.

V is disjoint from W.

Therefore, $\overline{V}$ is disjoint from V.

Therefore, $\overline{V} \subset U$.

Conversely, suppose given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset A$.

To prove that X is normal.

Let $U = X - B$ is an open set containing A.

By hypothesis, there exists an open set V containing A such that $\overline{V} \subset U$.

Therefore, the open set V and $X - \overline{V}$ are disjoint open set containing A and B respectively.

Also, given that the one point sets are closed in X.

Therefore, X is normal. □

Theorem 0.12.16. *(a) A subspace of a Hausdroff space is Hausdroff. A product of Hausdroff space is Hausdroff.*

(b) A subspace of a regular space is regular. A product of a regular space is regular.

Proof. (a) First let us prove the product of two hausdroff space is hausdroff.

Let X_1 and X_2 be two hausdroff spaces.

To prove $X_1 \times X_2$ is hausdroff space.

That is to prove $\forall x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $X_1 \times X_2$, $x \neq y$, there exists a neighbourhood U and V of (x_1, x_2) and (y_1, y_2) such that $U \cap V = \emptyset$.

Here $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in X_1$, $y_2 \in X_2$.

$x \neq y \Rightarrow (x_1, x_2) \neq (y_1, y_2)$

$\Rightarrow x_1 \neq y_1 \; or \; x_2 \neq y_2$.

We take $x_1 \neq y_1$.

Since X_1 is a hausdroff space, two point $x_1 \neq y_1$ of X_1, there exists a neighbourhood U_1 and U_2 of x_1 and y_1 such that $U_1 \cap U_2 = \emptyset$.

Consider $U_1 \times X_2$ and $U_2 \times X_2$.

Since U_1, U_2, X_2 are open, $U_1 \times X_2$ and $U_2 \times X_2$ are open.

Also, $(x_1, x_2) \in U_1 \times X_2$ and $(y_1, y_2) \in U_2 \times X_2$.

Since $U_1 \cap U_2 = \emptyset$, $(U_1 \times X_2) \cap (U_2 \times X_2) = \emptyset$.

Thus $U_1 \times X_2$ is a neighbourhood of x_1, x_2 and $U_2 \times X_2$ is a neighbourhood of y_1, y_2 with $(U_1 \times X_2) \cap (U_2 \times X_2) = \emptyset$.

Next to prove subspace of a hausdroff space is hausdroff.

Let X be a hausdroff space.

Let Y be a subspace of X.

To prove Y is hausdroff.

Let $y_1 \neq y_2$ be two points of Y. Then $y_1, y_2 \in X$.

Since X is hausdroff, there exists a neighbourhood U_{y_1} and U_{y_2} of y_1 and y_2 in X such that $U_{y_1} \cap U_{y_2} = \emptyset$.

Let $V_{y_1} = U_{y_1} \cap Y$ and $V_{y_2} = U_{y_2} \cap Y$.

Clearly, V_{y_1} and V_{y_2} are neighbourhood of y_1 and y_2 in Y.

$$\text{Also, } V_{y_1} \cap V_{y_2} = (U_{y_1} \cap Y) \cap (U_{y_2} \cap Y)$$
$$= (U_{y_1} \cap U_{y_2}) \cap Y$$

$= \emptyset \cap Y$

$= \emptyset.$

Therefore, Y is hausdroff.

(b) Let X be a regular space.

Let Y be a subspace of a regular space X.

Then one point sets are closed in Y.

Let x be a point of Y.

Let B be a closed set in Y not containing the point x.

Now, $\overline{B} \cap Y = B$ where $\overline{B}$ denotes the closure of B in X.

Therefore, $x \notin \overline{B}$.

So using regularity of X we can choose disjoint open sets U and V of X containing x and $\overline{B}$ respectively.

Then $U \cap Y$ and $V \cap Y$ are disjiont open sets containing X and B respectively.

Therefore, Y is regular.

That is the subspace of X is regular.

That is the subspace of X is regular.

Now, to prove product of a regular space is regular.

let $\{X_\alpha\}$ be a family of regular spaces.

Let $X = \Pi X_\alpha$.

By (a) part, X is hausdroff. So that one point sets are closed in X.

Let $x = (X_\alpha) \in X$.

Let U be a neighbourhood of x in X.

Choose a basis element ΠU_x about x contained in U.

Then U_α is a neighbourhood of x_α in X_α and each X_α is regular.

Choose for each α, the neighbourhood V_α of x_α such that $\overline{V_\alpha} \subset U_\alpha$. If it happens that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$.

Then $V = \Pi V_\alpha$ is a neighbourhood of x in X.

Since $\overline{V} \Pi \overline{V_\alpha}$.

By a theorem, it follows that, $\overline{V} \subset \Pi U_\alpha \subset U$.

That is $\overline{V} \subset U$.

Hence by lemma, X is regular.

That is ΠX_α is regular. □

0.13 Normal spaces

Theorem 0.13.1. *Every regular space with a countable basis is normal.*

Proof. Let X be a regular space with a countable basis $\mathscr{B}$.

To prove X is normal.

Let A and B be disjoint closed subsets of X. Each point x of A has a neighbourhood U not intersection B.

Using regularity choose a neigbourhood V of x whose closure lies in U.

Finally, choose an element of $\mathscr{B}$ contained in V.

By choosing such a basis element for each $x \in A$, we construct a countable covering of A by open sets whose closures do not intersect B.

Since this covering of A is countable, we can index it with the positive integers.

Let us denote it by $\{U_n\}$.

Similarly, we can choose a countable collection $\{V_n\}$ of open sets covers B such that each $\overline{V_n}$ does not intersect A.

The sets $U = \bigcup_{n \in z_+} U_n$ and

$V = \bigcup_{n \in z_+} V_n$.

Thus U and V are open sets containing A and B respectively but they need not be disjoint.

Now, we construct two disjoint open sets containing A and B respectively.

Define, $U_n^{'} = U_n - \bigcup_{i=1}^{n} \overline{V_i}$ and

$V_n^{'} = V_n - \bigcup_{i=1}^{n} \overline{U_i}$.

Each $U_n^{'}$ is the difference of a open set U_n and a closed set $\bigcup_{i=1}^{n} \overline{V_i}$.

Therefore, each $U_n^{'}$ is open.

Similarly, each $V_n^{'}$ is open.

Also, each $\overline{V_i}$ is disjoint from k.

$\{U_n^{'}/n \in Z_+\}$ is an open covering for A.

Simiolarly, $\{V_n^{'}/n \in Z_+\}$ is an open covering for B.

Let $U^{'} = \bigcup_{n \in Z_+} U_n^{'}$ and

$V^{'} = \bigcup_{n \in Z_+} V_n^{'}$.

Then $U^{'}$ and $V^{'}$ are open sets containing A and B.

Claim : $U^{'} \cap V^{'} = \emptyset$.

Suppose $U^{'} \cap V^{'} \neq \emptyset$.

Let $x \in U^{'} \cap V^{'}$

$\Rightarrow x \in U_j^{'} \cap V_k^{'}$ for some j and k.

Suppose $j \leq k$.

Now, $x \in U_j^{'} \Rightarrow x \in U_j$.

Also, $x \in V_k^{'} \Rightarrow x \notin \bigcup_{i=1}^{k} \overline{U_i}$

$\Rightarrow x \notin \overline{U_j}$.

Now, suppose that $j \geq k$.

Then we get $x \in V_k$ and $x \notin \overline{V_k}$. Which is a contradication.

Therefore, $U^{'} \cap V^{'} = \emptyset$. This proves the claim.

Therefore, $U^{'}$ and $V^{'}$ are disjoint open sets containing A and B.

Therefore, X is normal. □

Theorem 0.13.2. *Every metricable space is normal.*

Proof. Let X be a metricable space with metric d.

Let A and B be two disjoint closed subsets of X.

For each $a \in A$, choose $\epsilon_a > 0$ so that the open ball $B(a, \epsilon_a)$ does not intersect A.

Define, $U = \bigcup_{a \in A} B(a, \epsilon_{\frac{a}{2}})$ and $V = \bigcup_{b \in B} B(b, \epsilon_{\frac{b}{2}})$, U and V are open sets containing A and B respectively.

Claim: $U \cap V = \emptyset$.

For, if $U \cap V \neq \emptyset$.

Let $z \in U \cap V$. Therefore, $z \in B(a, \epsilon_{\frac{a}{2}}) \cap B(b, \epsilon_{\frac{b}{2}})$ for some $a \in A$ and $b \in B$.

We know that $d(a, b) \leq d(a, z) + d(z, b)$

$$< \epsilon_{\frac{a}{2}} + \epsilon_{\frac{b}{2}}$$

$$< \frac{\epsilon_a + \epsilon_b}{2}.$$

Now, if $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$

$\Rightarrow a \in B(b, \epsilon_b)$.

Also, if $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$

$\Rightarrow b \in B(a, \epsilon_a)$.

In both cases, we have a contradication.

Hence $U \cap V = \emptyset$.

Hence X is normal. □

Theorem 0.13.3. *Every compact Hausdroff space is normal.*

Proof. Let X be a compact hausdroff space.

To prove X is normal.

First let as prove X is regular.

For, if $x \in X$ and B is closed subset of X not containing x then, B is compact.

So that by a lemma, there exists disjoint open sets about x and B respectively.

Now we prove X is normal.

Let A and B be two disjoint closed subsets of X.

For each $a \in A$, choose disjoint open sets U_a and V_a containing A and B respectively.

This is possible, since X is regular. The collection $\{U_a\}$ covers A.

Since A is compact, it can be covered by finitily many collection of sets $\{U_{a_1}, U_{a_2}, \ldots, U_{a_m}\}$.

Define, $U = U_{a_1} \cup U_{a_2} \cup \ldots \cup U_{a_m}$ and $V = V_{a_1} \cap V_{a_2} \cap \ldots \cap V_{a_m}$.

Then U and V are disjoint open sets containing A and B respectively.

Hence X is normal. □

Theorem 0.13.4. *Every well ordered set X is normal in the order topology.*

Proof. Let X be well ordered set.

We assert that every interval of the form $(x, y]$ is open in X.

If X has a largest element and y is that element.

Then $(x, y]$ is just a basis element about y.

If y is not the largest element of X.

Then $(x, y]$ equals the open set (x, y') where y' is the immediate successor of y.

Now, let A and B be two disjoint closed sets in X.

Case (i) Assume for the moment neither A nor B contains the smallest element a_0 of X.

For $a \in A$, there exists a basis element about a disjoint from B.

It contains some interval of the form $(x, a]$.

Therefore, choose each $a \in A$ such an interval $(x_a, a]$ disjoint from B.

Choose an interval $(y_b, b]$ disjoint from A.

Define, $U = \bigcup_{a \in A}(x_a, a]$ and $V = \bigcup_{b \in B}(y_b, b]$.

Then $A \subset U$ and $B \subset V$ and U and V are open.

We assert that $U \cap V = \emptyset$.

For, if $U \cap V \neq \emptyset$.

Then $z \in U \cap V$

$\Rightarrow z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$.

Assume $a < b$.

If $a \leq y_b$.

Then the two intervals are disjoint while if $a > y_b$.

We have $a \in (y_b, b]$.

Contrary to the $(y_b, b]$ is disjoint from A, similar contradication occurs if $b < a$.

Case (ii) Now, assume A contains the smallest element a_0 of X.

The set $\{a_0\}$ is both open and closed in X.

The set $A - \{a_0\}$ and B are closed in X.

Therefore, by case (i), there exists disjoint open sets U and V containing $A - \{a_0\}$ and B respectively.

Therefore, $U \cup \{a_0\}$ and V are disjoint open sets containing A and B respectively.

Thus X is normal. □

Lemma 0.13.5. *(Urysohn Lemma)*

Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x)a$, for every $x \in A$ and $f(x) = b$, for every $x \in B$.

Proof. We will consider the only case of interval $[0, 1]$.

The general case follows from that one.

The first step of the proof is to construct.

Using normality a certain family U_p of open sets of X, indened by the rational numbers.

Then we use these sets to define the continuous function f.

Step 1: Let $P = \{p \in [0,1]/p\ is\ rational\}$.

Define for each $p \in P$, an open set U_p of X such that whenever $p < q$.

We have $\overline{U_p} \subset U_q$.

Since P is countable, we can use induction to define the set U_p.

Arrange the element of P in an infinite sequence in some way.

For convienience, let us suppose that the numbers 1 and 0 are the first two elements of the sequence.

First define $U_1 = X - B$ where A and B are closed subsets of X.

Second, because A is a closed set contain the open set U_1, by normality of X we can choose an open set U_0 such that $A \subset U_0$ and $\overline{U_0} \subset U_1$.

In general, P_n denote the set consisting of the first n- rational numbers in the sequence.

Suppose that U_p is defined for all rational numbers p belonging to P_n satisfing the condition $p < q \Rightarrow \overline{U_p} \subset U_q$ ——-(*)

Let r denote the next rational number in the sequence.

We wish to define U_r.

Consider the set, $P_{n+1} = P_n \cup \{r\}$.

It is a finite subset of the interval [0, 1] and its satisfies the simple order relation <.

We know that the finite simply order set, every element other than the smaller and the largest has a immediate predecessor and an immediate successor.

The number 0 is the smallest element and 1 is the largest element of P_{n+1}.

Therefore, r has an immediate predecessor p in P_{n+1} and an immediate successor q in $Pn+1$.

The sets U_p and U_q are already define and $\overline{U_p} \subset U_q$.

Using normality of X, we can find an open set U_r of X such that $\overline{U_p} \subset U_r$ and $\overline{U_r} \subset U_q$ ———— (1)

Claim : (*) holds for every pair of elements of P_{n+1}.

For, if both elements lieing P_n, (*) holds by induction hypothesis.

If one of them is r and other is an element s of P_n, then either $s \leq p$ or $s \geq q$.

If $s \leq p$, then $s < r$.

Since $s, p \in P_n$ by induction hypothesis $\overline{U_s} \subset U_p$.

That is $\overline{U_s} \subset U_p \subset \overline{U_p} \subset U_r$.

That is $\overline{U_s} \subset U_r$.

If $s \geq q$, $r < s$.

Since $q, s \in P_n$.

Then by induction hypothesis, $\overline{U_q} \subset U_s$.

That is $U_q \subset \overline{U_q} \subset U_s$.

That is $\overline{U_r} \subset U_q \subset \overline{U_q} \subset U_s$

$\Rightarrow \overline{U_r} \subset U_s$.

Therefore, eqation (*) ias true for every pair of elemevts in P_{n+1}.

Therefore, by induction for every $p \in P$ an open set U_p of X is defined such that whenever $p < q$, $\overline{U_p} \subset U_q$.

Step 2 : Now we define U_p for all rational in the interval $[0, 1]$ entened this definition to all rational numbers p in $\mathbb{R}$ by defining, $U_p = \emptyset$ if $p < 0$, $U_p = X$ if $p > 1$.

Then we have to prove for any rational numbers p and q in $\mathbb{R}$ whenever $p < q \Rightarrow$

$\overline{U_p} \subset U_q$.

Case(i) If p and q are two rational with $p < q$, then by step 1, $\overline{U_p} \subset U_q$.

Case (ii) If p and q are two rationals with $p \in [0, 1]$ and $q > p$.
Then U_p is defind by step 1 and $U_q = X$.
Therefore, $\overline{U_p} \subset U_q$.

Case(iii) If p and q are two rationals with $p < 0$ and $q \in [0, 1]$.
Then $U_p = \emptyset$ and U_q is defined by step 1,
$\Rightarrow \overline{U_p} = \overline{\emptyset} = \emptyset \subset U_q$.
THerefore, $\overline{U_p} \subset U_q$.

Csae(iv) If p and q are two rational numbers with $p < 0$ and $q > 1$.
Then $U_p = \emptyset,\ U_q = X$.
Therefore, $\overline{U_p} \subset U_q$.

It is still prove that for any pair of rational numbers p and q, $p < q \Rightarrow \overline{U_p} \subset U_q$.

Step 3:

Given a point $x \in X$.
Let us define $Q(x)$ to be the set of all rational numbers p such that the corresponding open sets U_p contains x.
That is $Q(x) = \{p/x \in U_p\}$.
This sets cointains no numbers ¡ 0.
Since no x is in U_P for $p > 1$.
Therefore, $Q(x)$ is bounded below and its greatest lower bound is a point of a interval $[0, 1]$.
Define $f(x) = \inf Q(x) = \inf\{p/x \in U_p\}$.

Step 4:

Claim 1: $f(x) = 0 \ \forall \ x \in A$ and $f(x) = 1, \ \forall \ x \in B$.

If $x \in A \Rightarrow x \in A \subset U_0 \subset \overline{U_1}$.

Now, $0 \leq p \Rightarrow \overline{U_0} \subset U_p$.

Therefore, $A \subset U_p$.

Hence $x \in A \Rightarrow x \in U_p$.

Therefore, $x \in U_p$ if $p \geq 0$.

That is $Q(x)$ contains all the rationals ≥ 0. Therefore, g.l.b of $Q(x) = 0$.

Therefore, $f(x) = 0, \ \forall \ x \in A$.

If $x \in B \Rightarrow x \notin U_1$

$$\Rightarrow x \notin \overline{U_p}$$

$$\Rightarrow x \notin U_p, \text{ if } p \leq 1.$$

Therefore, $Q(x)$ contains no rationals ≤ 1. Therefore, g.l.b of $Q(x) = 1$.

That is $f(x) = 1, \ \forall \ x \in B$.

Claim:2

Now, we show that f is continuous.

For this purpose we first prove the following elementary facts.

(1). $x \in \overline{U_r} \Rightarrow f(x) \leq r$

(2). $x \notin \overline{U_r} \Rightarrow f(x) \geq r$

To prove (1), Let $x \in \overline{U_r}$.

Then $x \in U_s$ for every $s > r$.

Therefore, $Q(x)$ contains all rational numbers $> r$.

So that by definition, we have $f(x) = \inf Q(x) \leq r$.

To prove (2)

Let $x \notin U_r$.

Then $x \notin U_s$ for every $s < r$.

Therefore, $Q(x)$ contains no rational numbers $< r$

So that by definition, we have $f(x) = \inf Q(x) \geq r$.

Now, we prove the continuity of f, $f : X \to \mathbb{R}$.

Given a point x_0 of X and an open interval (c, d) in $\mathbb{R}$ containing the point $f(x_0)$.

We wish to find a neigbourhood U of x_0 such that $f(U) \subset (c, d)$.

Choose rational numbers p and q such that $c < p < f(x_0) < q < d$.

We saaert that the open set, $U = U_q - \overline{U_p}$ is the desired neighbourhood of x_0.

First we note that $x_0 \in U_q$, for the fact that $f(x_0) < q \Rightarrow$ by condition (ii) that $x_0 \in U_q$ while the fact that $f(x_0) > p \Rightarrow$ by the condition (i) that $x_0 \notin \overline{U_p}$.

Second we show that $f(U) \subset (c, d)$.

Let $x \in U$, then $x \in U_q \subset \overline{U_q}$, so that $f(x) \leq q$ (by (1)) and $x \notin \overline{U_p}$ so that $f(x) \geq p$ (by (2)).

Thus $f(x) \in [p, q] \subset (c, d)$.

Therefore, $f(U) \subset (c, d)$.

Hence f is continuous.

$\square$

Unit 5

Banach spaces

0.14 The definition and some examples

Definition 0.14.1. *A linear space N is said to be a normed linear space if each vertex $x \in N$ there corresponds a real number, denoted by $\|x\|$ and called the norm of x, such that*

(1) $\|x\| \geq 0,$ *and* $\|x\| = 0 \Leftrightarrow x = 0$

(2) $\|x + y\| \leq \|x\| + \|y\|$*;*

(3) $\|\alpha x\| = |\alpha| \, \|x\|$.

A non-negative real number $\|x\|$ is a length of a vertex x. We can easily verify that the normed linear space N is a metric space with respect to the metric d defind by $d(x, y) = \|x - y\|$.

Definition 0.14.2. *A Banach space is a complete normed linear space.*

Result 0.14.3. *A function $\|\|: N \to R$ is continuous.*

Proof. First we prove that $|\|x\| - \|y\|| \leq \|x - y\|$.
We have $\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$.
Therefore $\|x\| - \|y\| \leq \|x - y\|$ ——————————(1)
Interchanging x and y we get, $\|y\| - \|x\| \|y - x\|$.
That is $-(\|x\| - \|y\|) \leq \|x - y\|$ ——————(2)
From (1) and (2) we get, $|\|x\| - \|y\|| \leq \|x - y\|$.
By the definition of continuvity, it is clear that $\|\|$ is continuous. □

Result 0.14.4. *Addition and scalar multiplication are jointly continuous.*

Proof. Suppose $x_n \to x$ and $y_n \to y$.
Now, $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$.
since $x_n \to x$.
since $x_n \to x$ and $y_n \to y$, the RHS converges to 0 and hence the LHS converges to 0. That is $(x_n + y_n) \to (x + y)$.
Suppose $\alpha_n \to \alpha$ and $x_n \to x$

$$\begin{aligned}
\text{Now, } \|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \\
&= \|\alpha_n (x_n - x) + (\alpha_n - \alpha) x\| \\
&\leq \|\alpha_n (x_n - x)\| + \|(\alpha_n - \alpha) x\| \\
&= |\alpha_n| \, \|x_n - x\| + |\alpha_n - \alpha| \, \|x\|
\end{aligned}$$

Since $\alpha_n \to \alpha$ and $x_n \to x$, the RHS converges to 0 and hence LHS converges to 0. That is $\alpha_n x_n \to \alpha x$. □

Theorem 0.14.5. *Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M denoted by $\|x + M\| = inf\{\|x + m\|\,; m \in M\}$ then N/M is a normed linear space. Further, if N is Banach space, then so in N/M.*

Proof. Part 1: Given N is a normed linear space and M is a closed linear space of N. To prove $(N/M, \|\|)$ is a normed linear space.
(1) Clearly $\|x + M\| \geq 0$ for every $x \in N$.
Suppose $x + M$ is a zero element of N/M. That is, $x + M = M$
$\Rightarrow x \in M$.

$$\begin{aligned}
\text{Now, } \|x + M\| &= inf\{\|x + m; m \in M\|\} \\
&= inf\{\|2\|\,; 2 \in M\}
\end{aligned}$$

$= 0.$

Hence $x + M = M \Rightarrow \|x + M\| = 0.$

Clearly, Suppose $\|x + M\| = 0$. That is $inf\{\|x + m; m \in M\|\} = 0.$

Then there exists a sequence(m_k) in M such that $\|x + m_k\| \to 0$

$\Rightarrow m_k \to -x$

$\Rightarrow -x \in M$

$\Rightarrow x \in M$

$\Rightarrow x + M = M,$ *the zero element of* $N/M.$

Therefore $\|x + M\| = 0 \Rightarrow x + M = M.$

(2) $\|(x + M) + (y + M)\| = \|(x + y) + M\|$

$$= inf\{\|x + y + m\|\,; m \in M\}$$

$$= inf\{\left\|x + y + m + m^{'}\right\|; m, m^{'} \in M\}$$

$$= inf\{\left\|(x + m) + (y + m^{'})\right\| \ ; m, m^{'} \in M\}$$

$$\leq inf\{\|x + m\| + \left\|y + m^{'}\right\|; m, m^{'} \in M\}$$

$$= inf\{\|x + m\|\,; m \in M\} + inf\{\left\|y + m^{'}\right\|;\ m^{'} \in M\}$$

$$= \|x + M\| + \|y + M\|.$$

(3) $\|\alpha(x + M)\| = \|\alpha x + M\|$

$$= inf\{\|\alpha x + m\|\,;\ m \in M\}$$

$$= inf\{\|\alpha(x + m)\|\,;\ m \in M\}$$

$$= inf\{|\alpha| \ \ \|x + m\|\,; m \in M\}$$

$$= |\alpha|\, inf\{\|x + m\|\,;\ m \in M\}$$

$$= |\alpha| \ \ \|x + M\|.$$

Hence N/M is a normed linear space.

Part 2: Given N is a Banach space. To prove N/M is complete.

That is, To prove any cauchy sequence in N/M has a convergent subsequence.

Clearly, it is possible to find a subsequence $\{x_n+M\}$ of the given cauchy sequence such that

$\|(x_1 + M) - (x_2 + M)\| < \frac{1}{2}$

$\|(x_2 + M) - (x_3 + M)\| < \frac{1}{4}$

$\vdots$

$\|(x_n + M) - (x_{n+1} + M)\| < \frac{1}{2^n}$.

To prove the sequence $\{x_n + M\}$ is convergent in N/M.

Choose any vector y_1 in $x_1 + M$ and choose any vector y_2 in $x_2 + M$ such that $\|y_1 - y_2\| < \frac{1}{2}$.

Again choose a vertor y_3 in $x_3 + M$ such that $\|y_2 - y_3\| < \frac{1}{4}$.

Continuing in this way, we obtain a sequence (y_n) in N such that $\|y_n - y_{n+1}\| < \frac{1}{2^n}$.

If $m < n$,

$$\begin{aligned}
\|y_m - y_n\| &= \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \ldots + (y_{n-1} - y_n)\| \\
&\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \ldots + \|y_{n-1} - y_n\| \\
&< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \frac{1}{2^{n-1}} \\
&< \frac{1}{2m-1}.
\end{aligned}$$

Therefore (y_n) is a cauchy sequence in N.

Since N is complete, there exists a vector y in N such that $y_n \to y$.

Now,
$$\begin{aligned}
\|(x_n + M) - (y + M)\| &= \|(x_n - y) + M\| \\
&= inf\{\|-y + y_n\|\,;\ y_n \in x_n + M\} \\
&\leq \|y_n - y\|.
\end{aligned}$$

Since $(y_n) \to y$, it follows that $x_n + M$ converges to $y + M$.

Therefore N/M is complete.

Hence N/M is Banach space. □

Example 0.14.6. *The space R and C- the real numbers and the complex numbers are normed linear spaces. The norm of a number x defind by $\|x\| = |x|$. Also R and C are Banach spaces.*

Example 0.14.7. *R^n and C^n of all n-tuples $x = (x_1, x_2, \ldots, x_n)$ of real and complex numbers can be made into normed linear space under the norm is $\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}}$. Also R^n and C^n are Banach spaces.*

Example 0.14.8. *let p be a real number such that $1 \leq p < \infty$. We denote by l_p^n, the space of all n-tuples $x = (x_1, x_2, \ldots, x_n)$, x_i are scalars. Show that l_p^n is a normed linear space under rhe norm defined by $\|x\|_p = [\sum_{i=1}^{n} |x_i|^p]^{\frac{1}{p}}$.*

Proof. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ and α be any scalar.

(1) Clearly, $\|x\|_p \geq 0$ (since each $|x_i| \geq 0$)

$$\|x\|_p = 0 \Leftrightarrow [\sum_{i=1}^{n} |x_i|^p]^{\frac{1}{p}} = 0$$
$$\Leftrightarrow [\sum_{i=1}^{n} |x_i|^p] = 0$$
$$\Leftrightarrow |x_i|^p = 0 \ for\ each\ i$$
$$\Leftrightarrow |x_i| = 0 \ for\ each\ i$$
$$\Leftrightarrow \ each\ x_i = 0$$
$$\Leftrightarrow x = (x_1, x_2, \ldots, x_n) = 0.$$

(2) $\|\alpha x\|_p = [\sum_{i=1}^{n} |\alpha x_i|^p]^{\frac{1}{p}}$
$$= [\sum_{i=1}^{n} |\alpha|^p |x_i|^p]^{\frac{1}{p}}$$
$$= |\alpha| [\sum_{i=1}^{n} |x_i|^p]^{\frac{1}{p}}$$
$$= |\alpha| \|x_p\|.$$

If $p = 1$

$\|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$

$\leq \sum_{i=1}^{n} [|x_i| + |y_i|]$

$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$

$= \|x\|_1 + \|y\|_1$

Therefore $\|x+y\|_1 \le \|x\|_1 + \|y\|_1$.

Thus the inequality holds when p = 1.

Consider $1 < p < \infty$.

Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$.

Now, $\|x+y\|_p = [\sum_{i=1}^{n} |x_i + y_i|^p]^{\frac{1}{p}}$

$$\|x+y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p$$
$$= \sum_{i=1}^{n} |x_i + y_i| \, |x_i + y_i|^{p-1}$$
$$\le \sum_{i=1}^{n} (|x_i| + |y_i|) \, |x_i + y_i|^{p-1}$$
$$= \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$
$$\le [\sum_{i=1}^{n} |x_i|^p]^{\frac{1}{p}} \left[\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right]^{\frac{1}{q}} + [\sum_{i=1}^{n} |y_i|^p]^{\frac{1}{p}} \left[\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right]^{\frac{1}{q}}$$
$$= \|x\|_p \left([\sum_{i=1}^{n} |x_i + y_i|^p]^{\frac{1}{q}}\right) + \|y\|_p \left([\sum_{i=1}^{n} |x_i + y_i|^p]^{\frac{1}{q}}\right)$$
$$= \|x\|_p \|x+y\|_p^{\frac{p}{q}} + \|y\|_p \|x+y\|_p^{\frac{p}{q}}$$
$$= \|x+y\|_p^{\frac{p}{q}} \left(\|x\|_p + \|y\|_p\right)$$

$\Rightarrow \|x+y\|_p^{(p-\frac{p}{q})} \le \|x\|_p + \|y\|_p$

$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p$.

Thus when $1 \le p < \infty$, $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

Therefore l_p^n is a normed linear space. □

Example 0.14.9. *The space l_p, consider a real number p with the property that $1 \le p < \infty$ and we denote by l_p, the space of all sequences $x = \{x_1, x_2, \ldots x_n, \ldots\}$ of scalars such that $\sum_{n=1}^{\infty} |x_p|^p < \infty$. Show that l_p is a normed linear space under the norm $\|x\|_p = [\sum_{n=1}^{\infty} |x_n|^p]^{\frac{1}{p}}$.*

Proof. Let $p = 1$

(1) $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$

Cleary, $\|x\|_1 \geq 0$, each $|x_i|$.

$\|x_1\| = 0 \Leftrightarrow \sum_{i=1}^{\infty} |x_i| = 0$

$\Leftrightarrow \ each \ \ |x_i| = 0$

$\Leftrightarrow \ each \ x_i = 0$

$\Leftrightarrow x = \{x_1, x_2, \ldots x_n, \ldots\} = 0.$

(2) $\|\alpha x\|_1 = \sum_{i=1}^{\infty} |\alpha x_i|$

$= |\alpha| \sum_{i=1}^{\infty} |x_i|$

$= |\alpha| \, \|x\|_1$

(3) $\|x + y\|_1 = \sum_{i=1}^{\infty} |x_i + y_i|$

$= \sum_{i=1}^{\infty} (|x_i| + |y_i|)$

$= \ lim_{n \to \infty} \sum_{i=1}^{n} (|x_i| + |y_i|)$

$= \ lim_{n \to \infty} \sum_{i=1}^{n} |x_i| + \ lim_{n \to \infty} \sum_{i=1}^{n} |y_i|$

$= \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i|$

$= \|x\|_1 + \|y\|_1$

Let $1 < p < \infty$

(1) $\|x\|_p = [\sum_{i=1}^{\infty} |x_i|^p]^{\frac{1}{p}} \geq 0$ (since each $|x|_i \geq 0$)

$\|x\|_p = 0 \Leftrightarrow [\sum_{i=1}^{\infty} |x_i|^p]^{\frac{1}{p}} = 0$

$\Leftrightarrow \sum_{i=1}^{\infty} |x_i|^p = 0$

$\Leftrightarrow |x_i|^p = 0$

$\Leftrightarrow |x_i| = 0 \ for \ each \ i$

$\Leftrightarrow \ each \ x_i = 0$

$\Leftrightarrow x = 0$

That is $\|x\|_p = 0 \Leftrightarrow x = 0$

(2) $\|\alpha x\|_p = [\sum_{i=1}^{\infty} |\alpha x_i|^p]^{\frac{1}{p}}$

$= [\sum_{i=1}^{\infty} |\alpha|^p |x_i|^p]^{\frac{1}{p}}$

$= |\alpha| [\sum_{i=1}^{\infty} |x_i|^p]^{\frac{1}{p}}$

$= |\alpha| \|x\|_p$

(3) $\|x+y\|_p = [\sum_{i=1}^{\infty} |x_i + y_i|^p]^{\frac{1}{p}}$

$= \ lim_{n\to\infty} [\sum_{i=1}^{n} |x_i + y_i|^p]^{\frac{1}{p}}$

$\leq \ lim_{n\to\infty} \left[(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}\right]$

$= \ lim_{n\to\infty} (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} + \ lim_{n\to\infty} (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}$

$= (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{\infty} |y_i|^p)^{\frac{1}{p}}$

$= \|x\|_p + \|y\|_p$

That is $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

□

Example 0.14.10. *Consider the linear space of all n-tuples $x = (x_1, x_2, \ldots, x_n)$ of scalars. Define the norm by $\|x\| = \ max\{|x_1|, |x_2|, \ldots, |x_n|\}$. This space is commonly denoted by l_∞^n and the symbol $\|x\|_\infty$is used for the norm. Show that l_∞^n is a normed linear space.*

Proof. (1) $\|x\|_\infty = \ max\{|x_1|, |x_2|, \ldots, |x_n|\}$

Clearly, $\|x\|_\infty \geq 0$

$\|x\|_\infty = 0 \Leftrightarrow \ max\{|x_1|, |x_2|, \ldots, |x_n|\} = 0$

$\Leftrightarrow |x_1| = 0, |x_2| = 0, \ldots, |x_n| = 0$

$\Leftrightarrow x_1 = 0, x_2 = 0, \ldots, x_n = 0$

$\Leftrightarrow (x_1, x_2, \ldots, x_n) = 0$

$\Leftrightarrow x = 0$

(2) $\|\alpha x\|_\infty = max\{|\alpha x_1|, |\alpha x_2|, \dots, |\alpha x_n|\}$

$= max\{|\alpha| |x_1|, |\alpha| |x_2|, \dots, |\alpha| |x_n|\}$

$= |\alpha| \; max\{|x_1|, |x_2|, \dots, |x_n|\}$

$= |\alpha| \|x\|_\infty$

(3) $\|x+y\|_\infty = max\{|x_1+y_1|, |x_2+y_2|, \dots, |x_n+y_n|\}$

$\leq max\{|x_1|+|y_1|, |x_2|+|y_2|, \dots, |x_n|+|y_n|\}$

$\leq max\{|x_1|, |x_2|, \dots, |x_n|\} + max\{|y_1|, |y_2|, \dots, |y_n|\}$

$= \|x\|_\infty + \|y\|_\infty$ □

Example 0.14.11. *Consider the linear space of all bounded sequences* $x = (x_1, x_2, \dots, x_n, \dots)$ *of all scalars. We define the norm* $\|x\| = sup|x|$*, and we denote the normed linear space is* l_∞*.*

Proof. (1) $\|x\| = sup|x_i|$

Clearly, $\|x\| > 0$

$\|x\| = 0 \Leftrightarrow sup|x_i| = 0$

$\Leftrightarrow |x_i| = 0, i = 1, 2, \dots$

$\Leftrightarrow x_i = 0, i = 1, 2, \dots$

$\Leftrightarrow x = 0$

(2) $\|\alpha x\| = sup|\alpha x_i|$

$= sup(|\alpha| |x_i|)$

$= |\alpha| \; sup|x_i|$

$= |\alpha| \|x\|$

(3) $\|x+y\| = sup(|x_i+y_i|)$

$= sup(|x_i| + |y_i|)$

$\leq \ sup\,|x_i| + \ sup\,|y_i|$

$= \|x\| + \|y\|$

Hence l_∞ is a normed linear space. □

0.15 Continuous linear transformations

Theorem 0.15.1. *Let N and N' be normed linear space and T a linear transformation of N into N' then the following conditions are equivalent*

(i) T is continuous;

(ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$;

(iii) there exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\,\|x\|$ for every $x \in N$;

(iv) if $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image $T(S)$ is a bounded set in N'.

Proof. $(i) \Rightarrow (ii)$

It is obvious that $(i) \Rightarrow (ii)$

Suppose that T is continuous at the origin.

Let $x_n \to x$

$\Rightarrow x_n - x \to 0$

$\Rightarrow T(x_n - x) \to T(0) = 0$

$\Rightarrow T(x_n - x) \to 0$

$\Rightarrow T(x_n) - T(x) \to 0$

$\Rightarrow T(x_n) \to T(x).$

Therefore T is continuous.

$(ii) \Rightarrow (i)$

Hence $(i) \Leftrightarrow (ii)$

To prove $(iii) \Rightarrow (ii)$

Suppose there exists a real number $K \geq 0$ such that $\|T(x)\| \leq |K| \, \|x\| \, \forall x \in N$.

Let $x_n \to 0$

Now $\|T(x_n)\| \leq |K| \, \|x_n\|$

Since $x_n \to 0, \; T(x_n) \to 0$

$\Rightarrow T$ is continuous at the origin

Hence $(iii) \Rightarrow (ii)$

To prove $(ii) \Rightarrow (iii)$

Let us assume that there is no K such that $\|T(x)\| \leq K \, \|x\| \, \forall x \in N$.

Then for any positive integer n, we can find a vector x_n such that $\|T(x_n)\| > n \, \|x_n\| \, \forall x \in N$

That is $\left\| T(\frac{x_n}{n\|x_n\|}) \right\| > 1$

Let $y_n = \frac{x_n}{n\|x_n\|}$

Clearly, $y_n \to 0$ $(since \; as \; n \to \infty, \frac{1}{n} \to 0)$

But $\|T(y_n)\| \nrightarrow 0$

Therefore T is not continuous at the origin.

Hence $(ii) \Leftrightarrow (iii)$

To prove $(iii) \Leftrightarrow (iv)$

Since $S = \{x : \|x\| \leq 1\}$,

$T(S) = \{T(x) : \|T(x)\| \leq 1\}$

Now $\|x\| \leq 1 \Rightarrow \|T(x)\| \leq K$

Therefore $T(S)$ contained in the closed sphere center on the origin of radius K.

Therefore $T(S)$ is a bounded set in N.

Thus $(iii) \Rightarrow (iv)$

Assume that $T(S)$ is bounded set in N, where $S = \{x : \|x\| \leq 1\}$.

Therefore $T(S)$ contained in the closed sphere center on the origin of radius K.

If x = 0, $T(x) = T(0) = 0$

Clearly $\|T(x)\| \leq K\|x\|$

If $x \neq 0$, $\frac{x}{\|x\|} \in S$

$T(\frac{x}{\|x\|}) \in T(S)$

Therefore $\left\|T(\frac{x}{\|x\|})\right\| \leq K$

$\Rightarrow \|T(x)\| \leq K\|x\|$

Therefore $(iv) \Rightarrow (iii)$

Hence $(iii) \Leftrightarrow (iv)$. □

Result 0.15.2. $\|T(x)\| \leq \|T\|\|x\|$

Proof. If $x = 0$, then $T(x) = 0$ and hence $\|T(x)\| \leq \|T\|\|x\|$.

If $x \neq 0 \in N$, then $\frac{x}{\|x\|} \in N$ and $\left\|\frac{x}{\|x\|}\right\| = 1$

$$\begin{aligned} \text{Now, } \|T(x)\| &= \left\|T\left(\frac{x}{\|x\|}\right)\|x\|\right\| \\ &= |\|x\||\left\|T\left(\frac{x}{\|x\|}\right)\right\| \\ &= \|x\|\left\|T\left(\frac{x}{\|x\|}\right)\right\| \\ &\leq \|x\|\|T\|. \end{aligned}$$

That is $\|T(x)\| \leq \|T\|\|x\|$

Hence for all $x \in N$, $\|T(x)\| \leq \|T\|\|x\|$. □

Notation:

$\mathscr{B}(N, N')$ is the set of all bounded (or continuous) linear transformations of N and N'.

Theorem 0.15.3. *If N and $N^{'}$ are normed linear spaces, then the set $\mathscr{B}(N, N^{'})$ of all continuous linear transformations of N and $N^{'}$ is itself a normed linear space with respect to the pointwise linear operations and the norm defind by* $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$. *Further if $N^{'}$ is a Banach space, then $\mathscr{B}(N, N^{'})$ is also a Banach space.*

Proof. We know that L, the set of all linear maps from N into $N^{'}$ is a linear space.

Now we will prove $\mathscr{B}(N, N^{'})$ is a subspace of L.

For, let $T_1, T_2 \in \mathscr{B}(N, N^{'})$

Then
$$\begin{aligned}\|(\alpha_1 T_1 + \alpha_2 T_2)(x)\| &= \|(\alpha_1 T_1)(x) + (\alpha_2 T_2)(x)\| \\ &= \|\alpha_1 T_1(x) + \alpha_2 T_2(x)\| \\ &\leq \|\alpha_1 T_1(x)\| + \|\alpha_2 T_2(x)\| \\ &= |\alpha_1| \|T_1(x)\| + |\alpha_2| \|T_2(x)\|\end{aligned}$$

Since $T_1, T_2 \in \mathscr{B}(N, N^{'})$, there exist $K_1, K_2 \geq 0$ such that $\|T_1(x)\| \leq K_1 \|x\|$ and $\|T_2(x)\| \leq K_2 \|x\| \ \forall x \in N$

Therefore
$$\begin{aligned}\|(\alpha_1 T_1 + \alpha_2 T_2)(x)\| &\leq |\alpha_1| K_1 \|x\| + |\alpha_2| K_2 \|x\| \\ &= (|\alpha_1| K_1 + |\alpha_2| K_2) \|x\| \\ &= K \|x\|\end{aligned}$$

That is $\|(\alpha_1 T_1 + \alpha_2 T_2)(x)\| \leq K \|x\|$

Thus $\alpha_1 T_1 + \alpha_2 T_2 \in \mathscr{B}(N, N^{'})$.

That is $\mathscr{B}(N, N^{'})$ is a subspace of L and hence $\mathscr{B}(N, N^{'})$ is a linear space.

Now we prove $\mathscr{B}(N, N^{'})$ is a normed linear space with the norm given by $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$.

Let $T \in \mathscr{B}(N, N^{'})$

Since $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$ *and* $\|T(x)\| \geq 0, \|T\| \geq 0$

Now $\|T\| = 0 \Leftrightarrow \sup\{\|T(x)\| : \|x\| \leq 1\} = 0$

$\Leftrightarrow \|T(x)\| = 0; \|x\| \leq 1$

$\Leftrightarrow T(x) = 0, x \in N, \|x\| \leq 1$

$\Leftrightarrow T = 0$

Let $T_1, T_2 \in \mathscr{B}(N, N^{'})$

$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

If α is a scalar and $T \in \mathscr{B}(N, N^{'})$

$$\|\alpha T\| = |\alpha| \, \|T\|$$

Therefore $\mathscr{B}(N, N^{'})$ is a normed linear space.

Let $N^{'}$ be a Banach space and $\{T_n\}$ be a cauchy sequence in $\mathscr{B}(N, N^{'})$.

If x is an arbitrary vector in N, then $\|T_m(x) - T_n(x)\| = \|(T_m - T_n)(x)\| \leq \|T_m - T_n\| \, \|x\|$

Since $\{T_n\}$ is a cauchy sequence, the RHS $\to 0$ as $n, m \to \infty$. Hence LHS $\to 0$ as $n, m \to \infty$.

Hence $\{T_n(x)\}$ is a cauchy sequence in $N^{'}$.

Since $N^{'}$ is a Banach space, $\{T_n(x)\}$ converges in $N^{'}$.

Now define $T(x) = \lim_{n\to\infty} T_n(x)$.

Claim : $T \in \mathscr{B}(N, N^{'})$.

For, let $x, y \in N$ and α, β are scalars.

Now, $T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$

$= \lim_{n\to\infty} \alpha T_n(x) + \lim_{n\to\infty} \beta T_n(y)$

$$= \alpha T(x) + \beta T(y).$$

Therefore T is linear.

$$\begin{aligned}\|T(x)\| \, \|\lim_{n\to\infty} T_n(x)\| &= \lim_{n\to\infty} \|T_n(x)\| \\ &\le \lim_{n\to\infty} \|T_n\| \, \|x\| \\ &\le \sup\{\|T_n\| \, \|x\|\} \\ &= (\sup \|T_n\|) \, \|x\| \text{————————————-(1)}\end{aligned}$$

Now, $|\|T_n\| - \|T_m\|| \le \|T_n - T_m\|$ which converges to 0 as $n, m \to \infty$. Therefore $\{\|T_n\|\}$ is a cauchy sequence of real numbers and hence convergent and bounded.

Hence there exists $K \ge 0$ such that $\sup \|T_n\| \le K$——————(2)

From (1) and (2) we get, $\|T(x)\| \le K \|x\| \;\; \forall \; x \in N$.

Thus $T \in \mathscr{B}(N, N')$.

Claim: $T_n \to T$.

Let $\epsilon > 0$ be given and let n_0 be a positive integer such that $n, m \ge n_0 \Rightarrow \|T_m - T_n\| < \epsilon$.

If $\|x\| \le 1$ and $m, n \ge n_0$, $\|T_m(x) - T_n\| = \|(T_m - T_n)(x)\|$

$$\begin{aligned}&\le \|T_m - T_n\| \, \|x\| \\ &\le \|T_m - T_n\| \\ &< \epsilon \text{ ————————(3)}\end{aligned}$$

Fix m and $n \to \infty$.

$\|T_m(x) - T_n(x)\| \to \|T_m(x) - T(x)\|$.

Now $\lim_{n\to\infty} \|T_m(x) - T_n(x)\| \le \epsilon$.

That is $\|T_m(x) - T(x)\| \le \epsilon, \;\; \|x\| \le 1 \; and \; for \; all \; m \ge n_0$.

Taking supremum on both sides, $\sup \|T_m(x) - T(x)\| \le \epsilon \; \forall \; m \ge n_0$

$\Rightarrow \|T_m - T\| \le \epsilon \; \forall \; m \ge n_0$

$\Rightarrow T_m \to T$

$\Rightarrow T_m \to T \in \mathscr{B}(N, N^{'})$.

$\mathscr{B}(N, N^{'})$ is a Banach space. □

Definition 0.15.4. *Let N be a normed linear space. A continuous linear transformation of N into itself is an operator on N. We denote the normed linear space of all operators on N by $\mathscr{B}(N)$ instead of $\mathscr{B}(N, N)$.*

Definition 0.15.5. *Let N and $N^{'}$ be normed linear space. The linear transformation $T : N \to N^{'}$ is said to be an isometric isomorphism if T is one-one and $\|T(x)\| = \|x\| \ \forall\, x \in N$. We say that N is isometrically isomorphic to $N^{'}$ if there exists an isometric isomorphism of N onto $N^{'}$.*

0.16 The Hahn - Banach Theorem

Definition 0.16.1. *Let N be a normed linear space. Now form the set of all continuous linear transformation of N into R or C accarding as N is real or complex. This set $\mathscr{B}(N, R)$ or $\mathscr{B}(N, C)$ it is denoted by N^* and is called conjugate space on N.*

The element of N^* are called continuous linear functionals or simply functionals.

If the norm of the functional $f \in N^*$ is defined by $\|f\| = \sup\{|f(x)| : \|x\| \leq 1\} = \inf\{K : K \geq 0 \textit{ and } |f(x)| \leq K\,\|x\| \ \forall\, x\}$ then N^* is a Banach space.

Lemma 0.16.2. *Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M and if $M_0 = M + [x_0]$*

is the linear subspace spanned by M and x_0, then f can be extented to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$

Proof. Case :1

Let N be a real normed linear space without loss of generality we may assume $\|f\| = 1$.

Since $x_0 \notin M$, each vector $y \in M_0$ is uniquely expressed in the form $y = x + \alpha x_0,\ x \in M$.

Define $f_0(y) = f_0(x + \alpha x_0)$

$= f_0(x) + \alpha f_0(x_0)$

$= f(x) + \alpha r_0$ where $r_0 = f_0(x_0)$ is a real number.

Clearly, f_0 is linear.

For, Let $y_1, y_2 \in M_0$ and $\beta, \gamma \in R$.

Then $y_1 = x_1 + \alpha_1 x_0$ and $y_2 = x_2 + \alpha_2 x_0,\ x_1, x_2 \in M$.

Now, $\beta y_1 + \gamma y_2 = \beta(x_1 + \alpha_1 x_0) + \gamma(x_2 + \alpha_2 x_0)$

$= (\beta x_1 + \gamma x_2) + (\beta\alpha_1 + \gamma\alpha_2)\, x_0$.

Now, $f_0(\beta y_1 + \gamma Y_2) = f_0\left[(\beta x_1 + \gamma x_2) + (\beta\alpha_1 + \gamma\alpha_2)x_0\right]$

$$= f_0(\beta x_1 + \gamma x_2) + (\beta\alpha_1 + \gamma\alpha_2) f_0(x)$$
$$= f(\beta x_1 + \gamma x_2) + (\beta\alpha_1 + \gamma\alpha_2) r_0$$
$$= \beta f(x_1) + \gamma f(x_2) + \beta\alpha_1 r_0 + \gamma\alpha_2 r_0$$
$$= \beta f(x_1) + \beta\alpha_1 r_0 + \gamma f(x_2) + \gamma\alpha_2 r_0$$
$$= \beta(f(x_1) + \alpha_1 r_0) + \gamma(f(x_2) + \alpha_2 r_0)$$
$$= \beta\left[f_0(x_1) + \alpha_1 f_0(x_0)\right] + \gamma\left[f_0(x_2) + \alpha_2 f_0(x_0)\right]$$
$$= \beta\left[f_0(x_1 + \alpha_1 x_0)\right] + \gamma\left[f_0(x_2 + \alpha_2 x_0)\right]$$
$$= \beta f_0(y_1) + \gamma f_0(y_2).$$

Therefore, f_0 is linear and hence f_0 is a linear extension of f.

Claim: $\|f_0\| = \|f\|$.

$$\begin{aligned}\|f_0\| &= \sup\{|f_0(x)| : x \in M_0,\ \ \|x\| \leq 1\} \\ &\geq \{|f_0(x)| : x \in M_0,\ \ \|x\| \leq 1\} \\ &= \sup\{|f(x)| : x \in M,\ \ \|x\| \leq 1\} \\ &= \|f\|.\end{aligned}$$

Therefore, $\|f_0\| \geq \|f\|$. ——————(1)

For any two vectors $x_1,\ x_2$ in M.

$$\begin{aligned}\text{We have } f(x_2) - f(x_1) &= f(x_2 - x_1) \\ &\leq |f(x_2 - x_1)| \\ &\leq \|f\|\,\|x_2 - x_1\| \\ &= \|x_2 - x_1\| \\ &= \|(x_2 + x_0) - (x_1 + x_0))\| \\ &= \|(x_2 + x_0) + (-(x_1 + x_0))\| \\ &\leq \|(x_2 + x_0)\| + \|(x_1 + x_0)\|.\end{aligned}$$

That is, $f(x_2) - f(x_1) \leq \|(x_2 + x_0)\| + \|(x_1 + x_0)\|$.

$\Rightarrow -f(x_1) - \|x_1 + x_0\| \leq -f(x_2) + \|x_2 + x_0\|$.

Since, this inequality holds for arbitrary $x_1, x_2, \ldots, x_n \in M$.

Now, $\sup_{x\in M}[-f(x) - \|x + x_0\|] \leq \inf_{x\in M}[-f(x) + \|x + x_0\|]$.

Choose r_0 to be any real number such that

$\sup_{x\in M}[-f(x) - \|x + x_0\|] \leq r_0 \leq \inf_{x\in M}[-f(x) + \|x + x_0\|]\ \ \forall\ x \in M$ ——(2).

With the choice of r_0, we will prove $\|f_0\| \leq \|f\|$.

Let $y = x + \alpha x_0$ be an arbitrary vector in M_0.

Replacing x by $\frac{x}{\alpha}$ in (2) we get,

$-f(\frac{x}{\alpha}) - \left\|\frac{x}{\alpha} + x_0\right\| \leq r_0 \leq -f(\frac{x}{\alpha}) + \left\|\frac{x}{\alpha} + x_0\right\|$ ———— (3).

If $\alpha > 0$, then $r_0 \leq -f(\frac{x}{\alpha}) + \left\|\frac{x}{\alpha} + x_0\right\|$

$\Rightarrow r_0 \leq -\frac{1}{\alpha}f(x) + \left|\frac{1}{\alpha}\right| \|x + \alpha x_0\|$

$\Rightarrow r_0 \leq -\frac{1}{\alpha}f(x) + \frac{1}{\alpha} \|x + \alpha x_0\|$

$\Rightarrow \alpha r_0 \leq -f(x) + \|x + \alpha x_0\|$

$\Rightarrow f(x) + \alpha r_0 \leq \|x + \alpha x_0\|$

$\Rightarrow f_0(x + \alpha x_0) \leq \|x + \alpha x_0\|$

$\Rightarrow f_0(y) \leq \|y\|$.

If $\alpha < 0$, then $r_0 \geq -f(\frac{x}{\alpha}) - \left\|\frac{x}{\alpha} + x_0\right\|$

$\Rightarrow r_0 \geq -\frac{1}{\alpha}f(x) - \left|\frac{1}{\alpha}\right| \|x + \alpha x_0\|$

$\Rightarrow r_0 \geq -\frac{1}{\alpha}f(x) + (-\frac{1}{\alpha}) \|x + \alpha x_0\|$

$\Rightarrow \alpha r_0 \leq -f(x) + \|x + \alpha x_0\|$

$\Rightarrow f(x) + \alpha r_0 \leq \|x + \alpha x_0\|$

$\Rightarrow f_0(x + \alpha x_0) \leq \|x + \alpha x_0\|$

$\Rightarrow f_0(y) \leq \|y\|$.

When $\alpha \neq 0$, $f_0(y) \leq \|y\| \ \forall \ y \in M$.

Replace y by -y we get,

$f_0(-y) \leq \|y\|$

$\Rightarrow -f_0(y) \leq \|y\|$

$\Rightarrow |f_0(y)| \leq \|y\|$ ————-(*).

Now, $\|f_0\| = \sup\{|f_0(y)| : y \in M_0, \ \|y\| \leq 1\}$

$\leq \sup\{\|y\| : y \in M_0, \ \|y\| \leq 1\}$

$= 1$

$= \|f\|$ ————-(4).

From (2) and (4) we get, $\|f_0\| = \|f\|$.

When $\alpha = 0$, $f = f_0$ and hence $\|f\| = \|f_0\|$.

Case ii

Let N be a complex normed linear space.

Hence f is complex valued functional defined on M such that $\|f\| = 1$.

Note that the complex linear space can be regarded as a real linear space by simply restricting the scalars to a real number.

Let $f(x) = g(x) + ih(x)$ where g and h are real for all $x \in M$.

Clearly, g and h real valued functionals defind on the real space.

Further, $|g(x)| \leq |f(x)|$

$$\leq \|f\| \, \|x\|.$$

Since f is bounded, g is bounded.

Similarly, h is also bounded.

Thus g and h are real valued functionals.

Now, $f(ix) = if(x)$

$\Rightarrow g(ix) + ih(ix) = i\,[g(x) + ih(x)]$

$\Rightarrow g(ix) + ih(ix) = ig(x) - h(x)$

$\Rightarrow h(x) = -g(ix)$.

Therefore, we can write $f(x) = g(x) - ig(ix)$.

By case (i), we can extend g to a real valud functional g_0 on the real space M_0 such that $\|g_0\| = \|g\|$.

Now, we define $f_0(x) = g_0(x) - ig_0(ix) \ \forall x \in M_0$.

Clearly, f_0 is a linear extension of f from M to M_0.

Thus f_0 is linear as a complex valued function defined on the complex space M_0.

Since f_0 is linear extension of f, we have $\|f_0\| \leq \|f\|$ ————-(5)

If $f_0(x)$ is real, then $f_0(x) = g_0(x)$.

Therefore, $\|f_0\| = \|g_0\|$

$$= \|g\|$$

$$= \sup\{|g(x)| : \|x\| \leq 1\}$$

$\leq \sup\{|f(x)| : \|x\| \leq 1\}.$

That is $\|f_0\| \leq \|f\|$ ————(6)

If $f_0(x)$ is complex, then we can write $f_0(x) = re^{i\theta}$ where $r > 0$ and $x \in M_0$ is arbitrary.

$$\begin{aligned}
\text{Now, } |f_0(x)| &= r \\
&= e^{-i\theta} r e^{i\theta} \\
&= e^{-i\theta} f_0(x) \\
&= f_0(e^{-i\theta} x) \\
&= g_0(e^{-i\theta} x) \\
&\leq \left|g_0(e^{-i\theta} x)\right| \\
&= \|g_0\| \left\|g_0(e^{-i\theta} x)\right\| \\
&= \|g_0\| \left|e^{-i\theta}\right| \|x\| \\
&= \|g_0\| \|x\|.
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, } |f_0(x)| &\leq \|g_0\| \|x\| \\
&= \|g\| \|x\| \\
&\leq \|f\| \|x\|.
\end{aligned}$$

Taking supremum, we get $\sup\{|f_0(x)| : x \in M_0,\ \|x\| \leq 1\} \leq \|f\|$.

That is $\|f_0\| \leq \|f\|$ ————(7).

From (5), (6) and (7) we get, $\|f_0\| = \|f\|$. □

Theorem 0.16.3. *(The Hahn Banach Theorem)*

Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. Then f can be extended to a functional f_0 defined on the whole space N such that $\|f_0\| = \|f\|$.

Proof. By Lemma, If $M_0 = N$ then there is nothing to prove.

Otherwise, let P denotes the set of all ordered pairs (g_λ, M_λ) where g_λ is a extension of f to the subspace $M_\lambda \supset M$ and $\|g_\lambda\| = \|f\|$.

Now, define the relation $\leq$ in P as follows:

$(g_\lambda, M_\lambda) \leq (g_\mu, M_\mu)$ where $M_\lambda \subset M_\mu$ and $g_\lambda \subset g_\mu$ on M_λ.

Clearly, the relation $\leq$ is partially ordered on P.

That is $(P, \leq)$ is a partially ordered set.

Clearly, P is nonempty, since $(f, M) \in P$.

Let $Q = \{(g_i, M_i)\}$ be a chain in P.

Define $\phi(x) = g_i(x) \; \forall \; x \in M$.

Now, $\bigcup M_i$ is a subspace of N and ϕ is well defined.

For, let $x, y \in \bigcup M_i$ and α and β be any scalars.

$\Rightarrow x \in M_i$ and $y \in M_j$ for some i and j.

Then either $M_i \subset M_j$ or $M_j \subset M_i$.

Without loss of generality, we assume $M_i \subset M_j$.

Then $x, y \in M_j$. Since M_j is a subspace of N, $\alpha x + \beta y \in M_j \subset \bigcup M_i$.

$\Rightarrow \alpha x + \beta y \in \bigcup M_i$.

Therefore, $\bigcup M_i$ is a subspace of N.

Let $x \in \bigcup M_i$ be the element such that $x \in M_i$ and $x \in M_j$.

Since $x \in M_i, \; \phi(x) = g_i(x)$ and $x \in M_j, \; \phi(x) = g_j(x)$.

Since Q is a chain, either g_i extends g_j or g_i extends g_j.

Hence $g_i(x) = g_j(x)$.

Therefore, ϕ is well defined.

Now, $(Q, \bigcup M_i)$ is an upper bound for P.

By Zorn's Lemma, $\exists \; (f_0, H)$ in P.

Claim: $N = H$.

Suppose $H \neq N$.

Then $\exists$ an eiement $x_0 \in N - H$ and by the lemma, f_0 can be extended to a functional g on $H_0 = H + [x_0]$ which contains H properly.

But this contradicts the maximality of (f_0, H) and hence we must have $H = N$ and f_0 is the required extension. $\square$

Theorem 0.16.4. *If N is a normed linear space and x_0 is a nonzero vector in N then $\exists$ a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.*

Proof. Let $M = \{\alpha x_0\}$ be the linear subspace of N spanned by x_0.

Define f on M by $f(\alpha x_0) = \alpha \|x_0\|$.

We show that f is a functional on M such that $\|f\| = 1$.

f is linear:

Let $y_1, y_2 \in M$ and α, β are scalars.

Then $y_1 = \gamma x_0$ and $y_2 = \delta x_0$.

$$\begin{aligned} \alpha y_1 + \beta y_2 &= \alpha\gamma x_0 + \beta\delta x_0 \\ &= (\alpha\gamma + \beta\delta)x_0. \end{aligned}$$

$$\begin{aligned} \text{Now, } f(\alpha y_1 + \beta y_2) &= f[(\alpha\gamma + \beta\delta)x_0] \\ &= \alpha\gamma + \beta\delta \|x_0\| \\ &= \alpha\gamma \|x_0\| + \beta\delta \|x_0\| \\ &= \alpha f(\gamma x_0) + \beta f(\delta x_0) \\ &= \alpha f(y_1) + \beta f(y_2). \end{aligned}$$

Therefore, f is linear.

f is bounded:

Let $y \in M$. Therefore, $y = \alpha x_0$ for some scalars.

$$\begin{aligned} \text{Now, } \|y\| &= \|\alpha x_0\| \\ &= |\alpha| \|x_0\| \end{aligned}$$

$|f(y)| = |f(\alpha x_0)|$

$= |\alpha \|x_0\||$

$= |\alpha| \, \|x_0\|$

$= \|y\|.$

Hence f is bounded.

It follows that f is functional on M.

Further, $\|f\| = \sup\{|f(y)| : y \in M, \ \ \|y\| \leq 1\}$

$= \sup\{\|y\| : y \in M, \ \ \|y\| \leq 1\}$

$= 1.$

Also $f(x_0) = f(1.x_0)$

$= \|x_0\|.$

Hence by Hahn Banach Theorem, f can be extended to a functional f_0 in N^* such that $\|f_0\| = \|f\|$.

Therefore, $f_0(x_0) = f(x_0)$ and $\|f_0\| = 1$.

That is $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. □

Theorem 0.16.5. *If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M then ∃ a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.*

Proof. Consider the natural map $T : N \to \frac{N}{M}$. Defined by $T(x) = x + M$.

Then T is a continuous linear transformation.

For, $x, y \in N$ and α, β be scalars.

Then $T(\alpha x + \beta y) = (\alpha x + \beta y) + M$

$= (\alpha x + M) + (\beta y + M)$

$= \alpha(x + M) + \beta(y + M)$

$= \alpha T(x) + \beta T(y).$

Therefore, T is linear.

Now, $\|T(x)\| = \|x + M\|$

$= \inf\{\|x + m : m \in M\|\}$

$\leq \|x + m\| \;\; \forall \; x \in M.$

In particular for m = 0, we have $\|T(x)\| \leq \|x\| \;\; \forall \; x \in N.$

Therefore, T is continuous.

Hence T is a continuous linear transformation.

If $m \in M$, then $T(m) = m + M$

$= M \; \forall m \in M$

$= 0.$

Therefore, $T(M) = 0$.

Since $x_0 \notin M$, we have $T(x_0) = x_0 + M \neq 0$ ————-(*).

Since by the theorem, $\exists$ a functional $f \in (\frac{N}{M})^*$ such that $f(x_0 + M) = \|x_0 + M\| \neq 0$.

That is $f(x_0 + M) \neq 0$.

Now we define $f_0(x) = f(T(x))$ when f_0 is a linear functional with the required properties.

For, $f_0(\alpha x + \beta y) = f(T(\alpha x + \beta y))$ where $x, y \in N$ and α, β are scalars,

$= f[\alpha T(x) + \beta T(y)]$

$= \alpha f(T(x)) + \beta f(T(x))$

$= \alpha f_0(x) + \beta f_0(y).$

Therefore, f_0 is linear.

Now, $|f_0(x)| = |f(T(x))|$

$\leq \|f\| \, \|T(x)\|$

$\leq \|f\| \, \|x\|.$

Therefore, f_0 is continuous.

Therefore, $f_0 \in N^*$.

Further, if $m \in M,\ f_0(m) = f(T(x_0))$

$$= f(x_0 + m)$$

$$\neq 0.$$

Thus $f_0(M) = 0$ and $f_0(x_0) \neq 0$. □

0.17 The Natural imbedding on N in N^{**}

Let N be a normed linear space. We know that the conjugate space N^* of N is also a normed linear space.

It is possible to form a conjugate space $(N^*)^*$ and we call it the second conjugate space of N.

Theorem 0.17.1. *Let N be a arbitrary normed linear space. Then each vector x in N induces a functional F_x on N^* defined by $F_x(f) = f(x)\ \forall f \in N^*$ such that $\|F_x\| = \|x\|$. Further, the mapping $J : N \to N^{**}$ defined by $J(x) = F_x\ \forall\ x \in N$ define an isometric isomorphism of N into N^{**}.*

Proof. First we show that F_x is a functional on N^*.

F_x is linear:

Let $f, g \in N^*$ and α, β be any scalars.

Now, $F_x(\alpha f + \beta g)(x) = (\alpha f + \beta g)(x)$

$$= \alpha f(x) + \beta g(x)$$

$$= \alpha F_x(f) + \beta F_x(g).$$

Therefore, F_x is linear.

F_x is bounded:

For any $f \in N^*$, we have

$|F_x(f)| = |f(x)|$

$\leq \|f\| \, \|x\|.$

That is $|F_x(f)| \leq \|f\| \, \|x\|$.

Hence F_x is a functional on N^*.

Claim : $\|F_x\| = \|x\|$.

$\|F_x\| = \sup\{|F_x(f)| : \|f\| \leq 1\}$

$= \sup\{|f(x)| : \|f\| \leq 1\}$

$\leq \sup\{\|f\| \, \|x\| : \|f\| \leq 1\}$

$\leq \|x\|$ ————(1)

To prove the reverse inequality, first we consider x = 0.

From (1) $\Rightarrow \|F_x\| = 0$

Also $\|x\| = 0$.

Therefore, $\|F_x\| = \|x\|$.

Let x be any non zero vector. Then by theorem, there exists a functional $f_0 \in N^*$ such that $f_0(x) = \|x\|$ and $\|f_0\| = 1$.

But $\|F_x\| = \sup\{|F_x(f)| : \|f\| = 1\}$

$= \sup\{|f(x)| : \|f\| = 1\}$.

Also $\|x\| = |f_0(x)| \leq \sup\{|f(x)| : \|f\| = 1\}$

$= \|F_x\|$

$\Rightarrow \|x\| \leq \|F_x\|$ ————-(2).

Now, we prove that J is a isometric.

That is to prove, J is a linear transformation as well as an isometric isomorphism.

J is linear:

Let $x, y \in N$ and α, β be any scalars.

Now, $F_{x+y}(f) = f(x+y)$

$$= f(x) + f(y)$$

$$= F_x(f) + F_y(f)$$

$$= (F_x + F_y)(f)$$

and $F_{\alpha x}(f) = f(\alpha x)$

$$= \alpha f(x)$$

$$= \alpha F_x(f) \ \forall \ f \in N^*.$$

Now, $J(x+y) = F_{x+y}$

$$= F_x + F_y$$

$$= J(x) + J(y)$$

and $J(\alpha x) = F_{\alpha x}$

$$= \alpha F_x$$

$$= \alpha J(x).$$

Therefore, J is linear.

J is an isometric:

Since $\|F_x\| = \|x\|$, we have $\|J(x)\| = \|x\|$.

For, $x, y \in N$, $\|J(x) - J(y)\| = \|F_x - F_y\|$

$$= \|F_{x-y}\|$$

$$= \|x - y\|.$$

Therefore, J is isomorphic.

Also $J_x - J_y = 0 \Rightarrow x - y = 0$

$\Rightarrow J_x = J_y \Rightarrow x = y$

Therefore, J is one to one. □